

# On the Solvability of a Class of Second Kind Integral Equations on Unbounded Domains

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We consider integral equations of the form  $\psi(x) = \phi(x) + \int_{\Omega} k(x, y)z(y)\psi(y) dy$  (in operator form  $\psi = \phi + K_z\psi$ ), where  $\Omega$  is some subset of  $\mathbf{R}^n$  ( $n \geq 1$ ). The functions  $k$ ,  $z$ , and  $\phi$  are assumed known, with  $z \in L_{\infty}(\Omega)$  and  $\phi \in Y$ , the space of bounded continuous functions on  $\bar{\Omega}$ . The function  $\psi \in Y$  is to be determined. The class of domains  $\Omega$  and kernels  $k$  considered includes the case  $\Omega = \mathbf{R}^n$  and  $k(x, y) = \kappa(x - y)$  with  $\kappa \in L_1(\mathbf{R}^n)$ , in which case, if  $z$  is the characteristic function of some set  $G$ , the integral equation is one of Wiener–Hopf type. The main theorems, proved using arguments derived from collectively compact operator theory, are conditions on a set  $W \subset L_{\infty}(\Omega)$  which ensure that if  $I - K_z$  is injective for all  $z \in W$  then  $I - K_z$  is also surjective and, moreover, the inverse operators  $(I - K_z)^{-1}$  on  $Y$  are bounded uniformly in  $z$ . These general theorems are used to recover classical results on Wiener–Hopf integral operators of H. Widom (*Inst. Hautes Études Sci. Publ. Math.* **44**, 1975, 191–240) and I. B. Simonenko (*Math. USSR-Sb.* **3**, 1967, 279–293), and generalisations of these results, and are applied to analyse the Lippmann–Schwinger integral equation.

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## 1. INTRODUCTION

We consider integral equations of the form

$$\psi(x) = \phi(x) + \int_{\Omega} k(x, y)z(y)\psi(y) dy, \quad x \in \bar{\Omega}, \quad (1)$$

where  $\Omega$  is an open subset of  $\mathbf{R}^n$  ( $n \geq 1$ ). The functions  $k: \bar{\Omega}^2 \rightarrow \mathbf{C}$ ,  $z \in X := L_{\infty}(\Omega)$ , and  $\phi \in Y := BC(\bar{\Omega})$ , the space of bounded continuous functions on  $\bar{\Omega}$ , are assumed known, and  $\psi \in Y$  is to be determined. Let

$k_x(y) = k(x, y)$ . We assume throughout that  $k$  is measurable and that  $k_x \in L_1(\Omega)$  for every  $x \in \bar{\Omega}$ , so that the integral in (1) is well-defined. We assume also that

- (A1)  $\sup_{x \in \bar{\Omega}} \int_{\Omega} |k(x, y)| dy < \infty$  and, through most of the paper, that
- (A2) for all  $x \in \bar{\Omega}$ ,

$$\int_{\Omega} |k(x, y) - k(x', y)| dy \rightarrow 0$$

as  $x' \rightarrow x$  with  $x' \in \bar{\Omega}$ .

Assumptions (A1) and (A2) ensure that  $K: X \rightarrow Y$  and is bounded [14], where  $K$  is the integral operator defined by

$$K\psi(x) = \int_{\Omega} k(x, y)\psi(y) dy, \quad x \in \bar{\Omega}. \tag{2}$$

In operator notation we abbreviate (1) as

$$\psi = \phi + K_z\psi = \phi + K(z\psi), \tag{3}$$

where  $K_z: Y \rightarrow Y$  is defined by

$$K_z\psi = K(z\psi), \quad \psi \in Y. \tag{4}$$

In the case when  $\Omega$  is unbounded we shall make, for most of the paper, the additional assumption that  $\Omega$  is the union of a bounded set  $\omega$  and translates of  $\omega$ , and that the integral operator  $K$  commutes with these translation operations. Precisely, the additional assumption is as follows where, as usual, for  $a \in \mathbf{R}^n$  and  $G \subset \mathbf{R}^n$ , we denote by  $G + a$  the translation of the set  $G$  by the vector  $a$ .

(A3) There exists  $\tau = \{a_1, \dots, a_m\} \subset \mathbf{R}^n$ , with  $m \leq n$ , and a bounded set  $\omega \subset \Omega$  such that:

- (i)  $\Omega = \omega \cup (\bigcup_{N \in \mathbf{Z}^m} \omega^{(N)})$ , where, for  $N = (n_1, \dots, n_m) \in \mathbf{Z}^m$ , we define  $\omega^{(N)} := \omega + \sum_{j=1}^m a_j n_j$ ;
- (ii)  $\omega^{(M)} \cap \omega^{(N)} = \emptyset$  for  $M, N \in \mathbf{Z}^m, M \neq N$ ;
- (iii)  $k(x + a, y + a) = k(x, y), x, y \in \bar{\Omega}, a \in \tau$ .

Note that (i) and (ii) imply that  $\Omega + a_j = \Omega, j = 1, \dots, m$ , so that (iii) makes sense. If  $\Omega$  is bounded then, with the understanding that  $\mathbf{Z}^0 := \emptyset$ , (A3) is trivially satisfied with  $\omega = \Omega, \tau = \emptyset$ , and  $m = 0$ , and (iii) then imposes no restriction on the kernel  $k$ .

An important case is that in which  $\Omega = \mathbf{R}^n$  and

$$k(x, y) = \kappa(x - y), \quad x, y \in \mathbf{R}^n, \quad (5)$$

with  $\kappa \in L_1(\mathbf{R}^n)$ . Then (A1)–(A3) are satisfied with various choices of  $\omega$ , for example,  $\omega = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n: 0 \leq x_j < 1, j = 1, \dots, n\}$  and  $\tau = \{e_1, \dots, e_n\}$ , with  $e_j$  the unit vector in the  $x_j$  direction. With  $z = \chi_G$ , the characteristic function of some measurable subset  $G \subset \mathbf{R}^n$ , Eq. (1) becomes the equation of Wiener–Hopf type,

$$\psi(x) = \phi(x) + \int_G \kappa(x - y)\psi(y) dy, \quad x \in \mathbf{R}^n. \quad (6)$$

We consider this case in some detail in Section 3.

Throughout, if (A3) is satisfied, define  $S_\tau = \emptyset$  if  $\tau = \emptyset$ , otherwise define

$$S_\tau := \left\{ \sum_{j=1}^m a_j n_j: N = (n_1, \dots, n_m) \in \mathbf{Z}^m \right\}. \quad (7)$$

For  $a \in S_\tau$ , define the translation operator  $T_a: X \rightarrow X$  by

$$T_a \psi(x) = \psi(x - a), \quad x \in \overline{\Omega}. \quad (8)$$

Then (and this is the point of assumption (A3)), if (A3) is satisfied it is easy to see that

$$T_a K = K T_a, \quad a \in S_\tau, \quad (9)$$

and that

$$\forall x \in \Omega, \exists a \in S_\tau \text{ such that } x - a \in \omega. \quad (10)$$

Let  $B(Y)$  denote the Banach space of bounded linear operators on  $Y$  and  $I$  the identity operator on  $Y$ . It is not difficult to show (see Theorem 2.11 below) that if (A1)–(A2) hold and  $\text{ess. sup}_{|x| \geq A} |z(x)| \rightarrow 0$  as  $A \rightarrow \infty$ , then  $I - K_z$  is Fredholm of index zero on  $Y$  so that

$$I - K_z \text{ injective} \Rightarrow I - K_z \text{ surjective}, \quad (I - K_z)^{-1} \in B(Y). \quad (11)$$

The main results of the paper are generalisations of (11) to cases where  $z$  does not vanish at infinity and  $I - K_z$  is not necessarily Fredholm. In such cases the result (11) does not hold in general for an individual  $z$ , but we show that it may apply in the following more general form: if  $W \subset X$  is

appropriately chosen then

$$I - K_z \text{ injective } \forall z \in W \Rightarrow I - K_z \text{ surjective } \forall z \in W,$$

$$\sup_{z \in W} \|(I - K_z)^{-1}\| < \infty.$$

In particular, in Section 2 we establish the following general result:

**THEOREM 2.8.** *Suppose that (A1)–(A3) are satisfied and that  $W \subset X$  is weak\* sequentially compact,  $T_a W = W$  for all  $a \in \tau$ , and  $I - K_z$  is injective for all  $z \in W$ . Then  $(I - K_z)^{-1}$  exists as an operator on the range space  $(I - K_z)Y$  for all  $z \in W$  and*

$$\sup_{z \in W} \|(I - K_z)^{-1}\| < \infty. \quad (12)$$

If also, for every  $z \in W$ , there exists a sequence  $(z_n) \subset W$  such that  $(z_n)$  converges weak\* to  $z$  and

$$\forall n, \quad I - K_{z_n} \text{ injective} \Rightarrow I - K_{z_n} \text{ surjective}, \quad (13)$$

then also  $I - K_z$  is surjective for each  $z \in W$  so that  $(I - K_z)^{-1} \in B(Y)$ .

The proof of the first part of this result generalises the usual contrapositive argument, for a compact operator  $K$ , that  $(I - K)^{-1}$  is bounded if  $I - K$  is injective, which is part of the Riesz theory. While the assumptions (A1)–(A3) do not ensure that  $K$  (or  $K_z$ ) is compact, (A1)–(A2) do ensure that  $K$  and  $K_z$  are locally compact. This, together with the translation invariance of  $\Omega$ ,  $K$ , and  $W$ , encapsulated in (A3), (9) and in the condition  $T_a W = W$ ,  $a \in \tau$ , is sufficient to complete the proof.

Various consequences of this result, and a generalisation of Theorem 2.8 presented as Theorem 2.7, are explored in Sections 2–4. For  $Q \subset \mathbf{C}$  let

$$L^Q := \{z \in X : \text{ess.range } z \subset Q\}. \quad (14)$$

We show in Section 2 that Theorem 2.8 can be applied with  $W = L^Q$ , provided  $Q$  is compact and convex, to obtain

**THEOREM 2.14.** *If (A1)–(A3) are satisfied,  $Q \subset \mathbf{C}$  is compact and convex, and  $I - K_z$  is injective for every  $z \in L^Q$  then  $I - K_z$  is surjective for every  $z \in L^Q$  and*

$$\sup_{z \in L^Q} \|(I - K_z)^{-1}\| < \infty.$$

A noteworthy special case of this result is that in which  $Q$  contains only a single point. Applying Theorem 2.14 in this case we obtain

**COROLLARY 2.15.** *If (A1)–(A3) are satisfied then the spectrum of  $K$  in  $B(Y)$  is the union of  $\{0\}$  and the set of eigenvalues of  $K$ .*

Further results, on the invertibility of  $I - K_z$  in weighted spaces of continuous functions, can be derived by combining the results of Section 2 with those in [8].

In Section 3 we consider the case, alluded to above, when (1) reduces to the Wiener–Hopf integral Eq. (6). Theorem 2.8 and its generalisation prove sufficiently powerful to prove a version of a result previously obtained by Widom [21] (and see Kozak [15]). In particular, we show that if  $G$  is a bounded open set of class  $C^1$ ,  $k(x, y) = \kappa(x - y)$  with  $\kappa \in L_1(\mathbf{R}^n)$ , and  $z_r := \chi_{rG}$ , for  $r > 0$ , where  $rG := \{rx : x \in G\}$ , then  $(I - K_{z_r})^{-1} \in B(Y)$  and is uniformly bounded for all sufficiently large  $r > 0$ , provided  $1 - \hat{\kappa}(\xi) \neq 0$ ,  $\xi \in \mathbf{R}^n$ , where  $\hat{\kappa}$  is the Fourier transform of  $\kappa$ ,

$$\hat{\kappa}(\xi) := \int_{\mathbf{R}^n} \exp(i\xi \cdot x) \kappa(x) dx, \quad \xi \in \mathbf{R}^n,$$

where  $\xi \cdot x$  is the usual Euclidean product of  $\xi$  and  $x$ . We also show a generalisation of this result and, as a corollary, the result that, if  $G \subset \mathbf{R}^2$  is a quadrant, then  $I - K_{\chi_G}$  is Fredholm of index zero provided  $1 - \hat{\kappa}(\xi) \neq 0$ ,  $\xi \in \mathbf{R}^2$ , i.e., provided the symbol of the operator  $I - K_{\chi_G}$  does not vanish. This result was previously obtained, using his local principle, for the case  $Y = L_p(\Omega)$ ,  $1 \leq p < \infty$ , by Simonenko [19], and by different methods in Strang [20] and Douglas and Howe [11], and is shown for the case  $p = \infty$  in Böttcher *et al.* [3].

In Section 4 we apply Theorem 2.14 to analyse the Lippmann–Schwinger integral equation [10]

$$u(x) = u^i(x) - \int_{\mathbf{R}^n} \Phi_{\lambda_0}(x - y)(\lambda_0^2 - \lambda^2(y))u(y) dy, \quad x \in \mathbf{R}^n. \quad (15)$$

This equation models time harmonic ( $e^{-i\omega t}$  time dependence,  $\omega > 0$ ) acoustic scattering of an incident wave  $u^i$  by an inhomogeneous medium of variable wave speed. In (15),  $\Phi_{\lambda_0}(x - y)$  is the free-field fundamental solution of the Helmholtz equation  $\Delta u + \lambda_0^2 u = 0$  (with  $\lambda_0$  a constant satisfying  $\Re \lambda_0 > 0$ ,  $\Im \lambda_0 \geq 0$ ), given by

$$\Phi_{\lambda_0}(x) = \frac{i}{4} \left( \frac{\lambda_0}{2\pi|x|} \right)^{n/2-1} H_{n/2-1}^{(1)}(\lambda_0|x|), \quad x \in \mathbf{R}^n, x \neq 0,$$

where  $H_{n/2-1}^{(1)}$  denotes the Hankel function of the first kind of order  $n/2 - 1$ . The function  $\lambda \in X$ , satisfying  $\Re \lambda > 0$ ,  $\Im \lambda \geq 0$ , has the interpretation that  $\lambda(x) = \omega/c(x)$  is the wave number and  $c(x)$  the wave speed at  $x \in \mathbf{R}^n$ .

We consider two cases, the first in which  $c(x) = c_0 := \omega/\lambda_0$  and  $\lambda(x) = \lambda_0$  outside a bounded open set  $\Omega$ . For this case the solvability of (15) is well-understood (see Colton and Kress [10] for an account of the three-dimensional case). We utilise Theorem 2.14 to prove the stability result, uniform in  $\lambda$ , that, provided  $\Lambda$  is a compact subset of  $\{z \in \mathbf{C}: \Re z > 0, \Im z \geq 0\}$ , then, for all  $\lambda \in L^\Lambda$ ,

$$\sup_{x \in \mathbf{R}^n} |u^s(x)| \leq C(\Omega, \Lambda) \sup_{x \in \Omega} |u^i(x)|, \quad (16)$$

where  $u^s := u - u^i$  is the scattered wave. We anticipate that this result, with the constant  $C(\Omega, \Lambda)$  independent of  $\lambda$ , will be of value in the analysis of inverse scattering and optimisation problems for (15).

For the case in which  $\lambda_0 - \lambda$  is not compactly supported we obtain the most complete results for the case in which  $\Im \lambda_0 > 0$  and  $\lambda \in L^\Lambda$  with  $\Lambda = \{re^{i\theta}: a \leq r \leq b\}$ , for some  $b > a > 0$  and  $0 < \theta < \pi$ . The solvability of (15) is considered in the weighted spaces  $Y(\eta)$ , defined, for  $\eta \in \mathbf{R}^n$ , by

$$Y(\eta) := \left\{ \psi \in C(\mathbf{R}^n): \|\psi\|_\eta := \sup_{x \in \mathbf{R}^n} |\psi(x)e^{-x \cdot \eta}| < \infty \right\}. \quad (17)$$

Rewriting (15) as an equation for the scattered field,

$$u^s(x) = F(x) - \int_{\mathbf{R}^n} \Phi_{\lambda_0}(x-y)(\lambda_0^2 - \lambda^2(y))u^s(y) dy, \quad x \in \mathbf{R}^n, \quad (18)$$

where

$$F(x) = - \int_{\mathbf{R}^n} \Phi_{\lambda_0}(x-y)(\lambda_0^2 - \lambda^2(y))u^i(y) dy, \quad x \in \mathbf{R}^n, \quad (19)$$

we show that, for every  $\eta \in \mathbf{R}^n$  with  $|\eta| < \epsilon := \min_{z \in \Lambda} \Im z = a \sin \theta$ . Eq. (18) has at most one solution in  $Y(\eta)$ . Further, if  $F \in Y(\eta)$ , then a solution  $u^s \in Y(\eta)$  exists and satisfies

$$\|u^s\|_\eta \leq C(\eta, \Lambda) \|F\|_\eta. \quad (20)$$

We point out that if  $u^i(x) = e^{\lambda_0 x_1}$  is a plane wave in the  $x_1$  direction and, for some  $\alpha \in \mathbf{R}$ ,  $\lambda = \lambda_0$  in the half-space  $x_1 < \alpha x_2$ , then  $F \in Y(\eta)$  for  $\eta = -\alpha \Im \lambda_0 u / \sqrt{1 + \alpha^2}$ , where  $u$  is the unit vector  $u = (\alpha e_1 + e_2) / \sqrt{1 + \alpha^2}$  and  $e_1$  and  $e_2$  are unit vectors in the  $x_1$  and  $x_2$  directions.

Clearly,  $|\eta| < \epsilon$  in this case provided that  $|\alpha| \sqrt{(\Im \lambda_0)^2 - \epsilon^2} < \epsilon$ .

Further applications of the results of Section 2 to wave scattering problems are contained in [7, 9].

The arguments used to prove our main results derive from those used in collectively compact operator theory [2]. These arguments were extended to investigate the finite section method for integral equations on the half line in Anselone and Sloan [5], and were further extended, in Chandler-Wilde [6], to deal with integral equations of the form (1) on the finite interval  $[a, b]$  and on the real line, with  $k(x, y) = \kappa(x - y)$  in the real line case. Our results and methods are closest to those of [6] in which a proof of Theorem 2.14 is given in the one-dimensional case when  $\Omega = \mathbf{R}$  and  $k(x, y) = \kappa(x - y)$  with  $\kappa \in L_1(\mathbf{R})$ .

## 2. THE MAIN RESULTS

For  $\phi \in X$  we denote by  $\|\phi\|$  the essential supremum norm,  $\|\phi\| := \text{ess. sup}_{x \in \Omega} |\phi(x)|$ . For a sequence  $(\psi_n) \subset X$  and  $\psi \in X$  we write  $\psi_n \rightarrow \psi$  if  $\|\psi_n - \psi\| \rightarrow 0$  as  $n \rightarrow \infty$ . Recalling that  $X$  can be identified with  $(L_1(\Omega))'$ , the dual space of  $L_1(\Omega)$ , we write  $\psi_n \xrightarrow{w^*} \psi$  if  $\psi_n$  converges to  $\psi$  in the weak\* topology on  $X$ , i.e., if

$$\int_{\Omega} \psi_n \phi \rightarrow \int_{\Omega} \psi \phi, \quad \forall \phi \in L_1(\Omega).$$

A useful characterisation of weak\* convergence in  $X$  is

$$\psi_n \xrightarrow{w^*} \psi \Leftrightarrow \sup_n \|\psi_n\| < \infty \quad \text{and} \quad \int_{\Omega} \psi_n \phi \rightarrow \int_{\Omega} \psi \phi, \quad \forall \phi \in C_0^\infty(\Omega), \quad (21)$$

where  $C_0^\infty(\Omega)$  is the set of compactly supported  $C^\infty$  functions defined on  $\Omega$ .

In the closed subspace  $Y \subset X$  we shall also be interested in convergence in the strict topology on  $Y$  of Buck [4]. For  $(\psi_n) \subset Y$ ,  $\psi \in Y$ , we write  $\psi_n \xrightarrow{s} \psi$  if  $\psi_n$  converges to  $\psi$  in this topology. Buck [4] shows that

$$\psi_n \xrightarrow{s} \psi \Leftrightarrow \sup_n \|\psi_n\| < \infty \quad \text{and} \quad \psi_n(x) \rightarrow \psi(x) \text{ uniformly on every compact subset of } \overline{\Omega}. \quad (22)$$

It is easy to see that

$$\phi_n \xrightarrow{w^*} \phi, \psi_n \xrightarrow{s} \psi \Rightarrow \phi_n \psi_n \xrightarrow{w^*} \phi \psi \tag{23}$$

and that

$$\psi_n \xrightarrow{s} \psi \Rightarrow \|\psi\| \leq \sup_n \|\psi_n\|. \tag{24}$$

We will need the following continuity and compactness properties of the integral operator  $K$  with respect to the above topologies.

LEMMA 2.1. *If  $k$  satisfies (A1) then  $\phi_n \xrightarrow{w^*} \phi \Rightarrow K\phi_n \xrightarrow{w^*} K\phi$ .*

*Proof.* Let  $K'$  denote the transpose operator with kernel  $k(y, x)$  and note that (A1) ensures that  $K': L_1(\Omega) \rightarrow L_1(\Omega)$  and is bounded [14]. If  $\phi_n \xrightarrow{w^*} \phi$  and  $\psi \in L_1(\Omega)$  then, reversing the order of integration using Fubini's theorem,

$$\int_{\Omega} (K\phi_n)\psi = \int_{\Omega} \phi_n(K'\psi) \rightarrow \int_{\Omega} \phi(K'\psi) = \int_{\Omega} (K\phi)\psi,$$

so that  $K\phi_n \xrightarrow{w^*} K\phi$ . ■

LEMMA 2.2. *If  $B \subset X$  is bounded and  $k$  satisfies (A1) and (A2) then every sequence  $(\phi_n) \subset KB$  has a strictly convergent subsequence.*

*Proof.* Using the Arzela–Ascoli theorem and (22) it is not difficult to see that it is sufficient to show that  $KB$  is bounded and equicontinuous, i.e., that  $\sup_{\psi \in B} \|K\psi\| < \infty$  and that, for all  $x \in \Omega$ ,  $K\psi(x') \rightarrow K\psi(x)$  as  $x' \rightarrow x$ , uniformly in  $\psi$  for  $\psi \in B$  (cf. [5, 16]). But this follows easily from (A1) and (A2). ■

COROLLARY 2.3. *If  $k$  satisfies (A1) and (A2) then  $\phi_n \xrightarrow{w^*} \phi \Rightarrow K\phi_n \xrightarrow{s} K\phi$ .*

*Proof.* We note that

$$\psi_n \xrightarrow{s} \psi \Leftrightarrow \text{every subsequence of } (\psi_n) \text{ has a subsequence converging to } \psi, \tag{25}$$

so that our result will follow if we can show that  $\phi_n \xrightarrow{w^*} \phi \Rightarrow K\phi_{n_m} \xrightarrow{s} K\phi$ , for some subsequence  $(\phi_{n_m})$ . But it follows from (21) and Lemma 2.2, that, for some subsequence  $(\phi_{n_m})$ ,  $K\phi_{n_m} \xrightarrow{s} \psi \in Y$ . From Lemma 2.1 we have that  $K\phi_{n_m} \xrightarrow{w^*} K\phi$  so that  $\psi = K\phi$ . ■

These preliminaries are sufficient to prove our first main result. In this theorem we use the notation  $W^\times := W_1 \times W_2 \times \dots$  to denote the set of sequences  $(\phi_n)$  such that  $\phi_1 \in W_1, \phi_2 \in W_2, \dots$

**THEOREM 2.4.** *Suppose that  $W$  and  $W_n$ ,  $n \in \mathbf{N}$ , are subsets of  $X$  and that:*

- (i) *Assumptions (A1)–(A3) are satisfied;*
- (ii) *For every  $n \in \mathbf{N}$ ,  $T_a W_n = W_n$ ,  $a \in \tau$ ;*
- (iii) *Every sequence  $(z_n) \in W^\times := W_1 \times W_2 \times \dots$  has a subsequence  $(z_{n_m})$  such that  $z_{n_m} \xrightarrow{w^*} z \in W$ ;*
- (iv)  *$I - K_z$  is injective for every  $z \in W$ .*

*Then, for some  $N \in \mathbf{N}$  and all  $n \geq N$  and  $z \in W_n$ ,  $I - K_z$  is injective so that  $(I - K_z)^{-1}$  exists as an operator on the range space  $(I - K_z)Y$ . Moreover*

$$\sup_{z \in W_n, n \geq N} \|(I - K_z)^{-1}\| < \infty.$$

*Proof.* Suppose that the conclusions are false. Then there exists  $(z_n) \in W^\times$  and  $(\phi_n) \subset Y$  with  $\|\phi_n\| = 1$  for each  $n$  such that  $\phi_n - K_{z_n} \phi_n \rightarrow 0$ . By (10), which follows from assumption (A3), we can choose a sequence  $(a_n) \subset S_\tau$  ( $S_\tau$  defined by (7)) such that

$$\sup_{x \in \omega} |\psi_n(x)| \geq \frac{1}{2},$$

where  $\psi_n := T_{a_n} \phi_n$ . By (9),

$$T_{a_n} K_{z_n} \phi_n = K T_{a_n} (z_n \phi_n) = K(\tilde{z}_n \psi_n) = K_{\tilde{z}_n} \psi_n,$$

where  $\tilde{z}_n := T_{a_n} z_n \in W_n$ , since  $T_a W_n = W_n$  for all  $a \in S_\tau$ . Thus

$$\psi_n - K(\tilde{z}_n \psi_n) \rightarrow 0.$$

By Lemma 2.2 and (iii), there exist subsequences  $(\tilde{z}_{n_m})$  and  $(\psi_{n_m})$  such that  $\tilde{z}_{n_m} \xrightarrow{w^*} z \in W$  and  $K(\tilde{z}_{n_m} \psi_{n_m}) \xrightarrow{s} \psi \in Y$ . Thus  $\psi_{n_m} \xrightarrow{s} \psi$  and, by (23),  $\tilde{z}_{n_m} \psi_{n_m} \xrightarrow{w^*} z\psi$  whence, by Corollary 2.3,  $K(\tilde{z}_{n_m} \psi_{n_m}) \xrightarrow{s} K(z\psi) = K_z \psi$ . Thus  $\psi = K_z \psi$  and, since  $I - K_z$  is injective,  $\psi = 0$ . But this is a contradiction since  $\sup_{x \in \omega} |\psi_n(x)| \geq \frac{1}{2} \Rightarrow \sup_{x \in \omega} |\psi(x)| \geq \frac{1}{2}$ . ■

The next two results (cf. [5; 6, Theorem 4.4]) will be used to establish conditions, additional to those in Theorem 2.4, which will ensure that also  $I - K_z$  is injective so that  $(I - K_z)^{-1} \in B(Y)$  for each  $z \in W$ .

**LEMMA 2.5.** *Suppose that conditions (A1) and (A2) are satisfied, that  $z_n \xrightarrow{w^*} z$  and  $\phi_n \xrightarrow{s} \phi$ , that  $I - K_z$  is injective, and that  $(\psi_n) \subset Y$  is bounded and satisfies  $\psi_n = \phi_n + K_{z_n} \psi_n$ , for each  $n$ . Then  $\phi \in (I - K_z)Y$ , so that  $\psi := (I - K_z)^{-1} \phi$  exists, and  $\psi_n \xrightarrow{s} \psi$ .*

*Proof.* Since  $(z_n \psi_n)$  is bounded, by Lemma 2.2, for some subsequences  $(z_{n_m})$ ,  $(\psi_{n_m})$ , and  $(\phi_{n_m})$ ,  $K(z_{n_m} \psi_{n_m}) + \phi_{n_m} \xrightarrow{s} \psi \in Y$ . Thus  $\psi_{n_m} \xrightarrow{s} \psi$  and, by (23) and Corollary 2.3,  $z_{n_m} \psi_{n_m} \xrightarrow{s} z \psi$  and  $K(z_{n_m} \psi_{n_m}) \xrightarrow{s} K(z \psi) = K_z \psi$ . Thus  $\psi = K_z \psi + \phi$ , so that  $\phi \in (I - K_z)Y$  and  $\psi = (I - K_z)^{-1} \phi$ .

By the same argument we can show that every subsequence of  $(\psi_n)$  has a subsequence converging strictly to  $\psi$ . Thus, by (25),  $\psi_n \xrightarrow{s} \psi$ . ■

**THEOREM 2.6.** *Suppose that conditions (A1) and (A2) are satisfied, that  $z_n \xrightarrow{w^*} z$ , that  $I - K_z$  is injective, and that  $(I - K_{z_n})^{-1} \in B(Y)$  for each  $n$ , with  $C := \sup_n \|(I - K_{z_n})^{-1}\| < \infty$ . Then  $(I - K_z)^{-1} \in B(Y)$ ,  $\|(I - K_z)^{-1}\| \leq C$ , and, for all  $\phi \in Y$ ,  $(I - K_{z_n})^{-1} \phi \xrightarrow{s} (I - K_z)^{-1} \phi$ .*

*Proof.* Since  $I - K_z$  is injective,  $(I - K_z)^{-1}$  exists as an operator on  $(I - K_z)Y$ .

Let  $\phi \in Y$  and define the sequence  $(\psi_n) \subset Y$  by  $\psi_n := (I - K_{z_n})^{-1} \phi$ . Then  $(\psi_n)$  is bounded and  $\psi_n = K(z_n \psi_n) + \phi$ , so that, by Lemma 2.5,  $\phi \in (I - K_z)Y$  and  $\psi_n \xrightarrow{s} (I - K_z)^{-1} \phi$ . Thus  $I - K_z$  is surjective and  $(I - K_{z_n})^{-1} \phi \xrightarrow{s} (I - K_z)^{-1} \phi$  for each  $\phi \in Y$ . From this it follows, by (24), that  $\|(I - K_z)^{-1}\| \leq C$ . ■

The following result now follows straightforwardly from Theorems 2.4 and 2.6:

**THEOREM 2.7.** *Suppose that the conditions of Theorem 2.4 are satisfied and that, in addition, for every  $z \in W$ , there exists  $(z_n) \in W^\times$  such that  $z_n \xrightarrow{w^*} z$  and*

$$\forall n, \quad I - K_{z_n} \text{ injective} \Rightarrow I - K_{z_n} \text{ surjective.}$$

*Then, for all  $z \in W$ ,  $(I - K_z)^{-1} \in B(Y)$  and, for all sufficiently large  $N$ ,*

$$\sup_{z \in W} \|(I - K_z)^{-1}\| \leq \sup_{z \in W_n, n \geq N} \|(I - K_z)^{-1}\| < \infty.$$

Theorem 2.8 is the special case of Theorems 2.4 and 2.7 which arises if we take  $W_n = W$ , for  $n \in \mathbf{N}$ .

To apply the above results, conditions for the weak\* sequential compactness of  $W \subset X$  will be useful. We point out first that, since  $L_1(\Omega)$  is separable, by the Banach–Alaoglu theorem [18], every closed ball in  $X$  is weak\* sequentially compact, so that

$$W \text{ weak}^* \text{ sequentially compact} \Leftrightarrow W \text{ bounded, weak}^* \text{ sequentially closed.} \tag{26}$$

Further, defining for  $A > 0$ ,  $\Omega_A := \{x \in \Omega: |x| < A\}$  and  $W^{(A)} := \{\psi|_{\Omega_A}: \psi \in W\}$ , it follows from (21) that

$$\begin{aligned} &W \text{ bounded, weak}^* \text{ sequentially closed} \\ &\Leftrightarrow W \text{ bounded (in } X), W^{(A)} \text{ weakly sequentially closed} \\ &\quad \text{in } L_2(\Omega_A) \forall A > 0. \end{aligned} \tag{27}$$

Further, if  $V \subset L_2(\Omega_A)$  is convex, then  $V$  is weakly sequentially closed iff it is closed [1], so that, if  $W \subset X$  is convex,

$$\begin{aligned} &W \text{ weak}^* \text{ sequentially compact} \\ &\Leftrightarrow W \text{ bounded in } X, W^{(A)} \text{ closed in } L_2(\Omega_A) \forall A > 0. \end{aligned} \tag{28}$$

Conditions on  $k$  and  $z$  which ensure that  $I - K_z$  injective  $\Rightarrow I - K_z$  surjective are also required. These are supplied in Theorems 2.10 and 2.11 below. As a preliminary we first study the invertibility of  $I - K_z$  in subspaces of  $X$  consisting of functions periodic on  $\Omega$ .

Suppose that (A3) is satisfied and, for  $l \in \mathbf{N}$ , let  $S_\tau^{(l)} := \{3^{l-1}a: a \in S_\tau\}$ , so that  $S_\tau^{(l)} \subseteq S_\tau$ , with equality if  $l = 1$ , and define the closed subspaces  $X_\tau^{(l)}, Y_\tau^{(l)} \subset X$ , by  $X_\tau^{(l)} := \{\psi \in X: T_a \psi = \psi, a \in S_\tau^{(l)}\}$  and  $Y_\tau^{(l)} := Y \cap X_\tau^{(l)}$ . Clearly, if  $\psi \in X_\tau^{(l)}$  and  $a \in S_\tau^{(l)}$ , then  $\psi(x + a) = \psi(x)$ ,  $x \in \bar{\Omega}$ . If  $S_\tau = \emptyset$  (the case if  $\Omega$  is bounded) then  $X_\tau^{(l)} = X$ .

For  $N = (n_1, \dots, n_m) \in \mathbf{Z}^m$  define  $|N| := \max(|n_1|, \dots, |n_m|)$ . If (A3) is satisfied, let  $\omega_1 := \omega$  and define, for  $l = 2, 3, \dots$ ,

$$\omega_l := \bigcup_{N \in \mathbf{Z}^m, |N| \leq (3^{l-1} - 1)/2} \omega^{(N)}.$$

For  $N \in \mathbf{Z}^m$  and  $l \in \mathbf{N}$  define

$$\omega_l^{(N)} := \omega_l + \sum_{j=1}^m 3^{l-1} a_j n_j,$$

and note that, by (A3),  $\Omega = \omega_l \cup (\bigcup_{N \in \mathbf{Z}^m} \omega_l^{(N)})$  and  $\omega_l^{(M)} \cap \omega_l^{(N)} = \emptyset$  for  $M, N \in \mathbf{Z}^m$ ,  $M \neq N$ . For  $l \in \mathbf{N}$ , define the periodic extension operator  $P_\tau^{(l)}: Y \rightarrow X_\tau^{(l)}$  by

$$P_\tau^{(l)}\psi(x) := \psi\left(x - \sum_{j=1}^m 3^{l-1} a_j n_j\right), \quad x \in \omega_l^{(N)}, \tag{29}$$

for  $N = (n_1, \dots, n_m) \in \mathbf{Z}^m$ .

LEMMA 2.9. *If (A1)–(A3) are satisfied then, for  $l \in \mathbf{N}$ ,  $K: X_\tau^{(l)} \rightarrow Y_\tau^{(l)}$  and is compact.*

*Proof.* Clearly, in view of the definition of  $X_\tau^{(l)}$  and (9),  $K: X_\tau^{(l)} \rightarrow Y_\tau^{(l)}$ . Further, this mapping is compact since, if  $(\psi_n) \subset X_\tau^{(l)}$  is bounded, then, by Lemma 2.2,  $(K\psi_n)$  has a strictly convergent subsequence which, by (22), is uniformly convergent on  $\overline{\omega}_l$ . But, since  $(K\psi_n) = (P_\tau^{(l)}K\psi_n)$ , it follows that this subsequence is uniformly convergent on the whole of  $\overline{\Omega}$ . ■

THEOREM 2.10. *If  $k$  satisfies (A1)–(A3) and  $z(x) = \lambda \in \mathbf{C}$  for almost all  $x \in \Omega$  then*

$$I - K_z \text{ injective} \Rightarrow I - K_z \text{ surjective}, \quad (I - K_z)^{-1} \in B(Y).$$

*Proof.* We have that, for  $l \in \mathbf{N}$ ,

$$\begin{aligned} I - K_z \text{ injective on } Y &\Rightarrow I - K_z \text{ injective on } Y_\tau^{(l)} \\ &\Rightarrow I - K_z \text{ surjective on } Y_\tau^{(l)}, \end{aligned}$$

since  $K_z$  is compact on  $Y_\tau^{(l)}$ , by Lemma 2.9. Thus, if  $I - K_z$  is injective (on  $Y$ ), then  $Y_\tau^{(l)} \subset (I - K_z)Y$  for every  $l \in \mathbf{N}$ .

Let  $W = \{z\}$ . If  $I - K_z$  is injective then  $W$  satisfies the conditions of the first part of Theorem 2.8, so that  $(I - K_z)^{-1}$  exists as an operator on the range space  $(I - K_z)Y$  and  $(I - K_z)^{-1}$  is bounded.

So suppose that  $I - K_z$  is injective, let  $\phi \in Y$ , note that  $K_z P_\tau^{(n)}\phi \in (I - K_z)Y$  by the above and Lemma 2.9, and define  $\psi_n := \phi + (I - K_z)^{-1}K_z P_\tau^{(n)}\phi$ , for  $n \in \mathbf{N}$ . Then  $(\psi_n)$  is bounded and  $\psi_n = K_z \psi_n + \phi_n$ , where  $\phi_n := \phi + K_z(P_\tau^{(n)}\phi - \phi) \xrightarrow{s} \phi$ , by Corollary 2.3. It follows from Lemma 2.5 that  $\phi \in (I - K_z)Y$ . Thus  $I - K_z$  is surjective and  $(I - K_z)^{-1} \in B(Y)$ . ■

THEOREM 2.11. *If  $k$  satisfies (A1) and (A2),  $z_1, z_2 \in X$ , and  $\text{ess.sup}_{|x| \geq A} |z_1(x) - z_2(x)| \rightarrow 0$  as  $A \rightarrow \infty$ , then*

$$I - K_{z_1} \text{ Fredholm} \Leftrightarrow I - K_{z_2} \text{ Fredholm}$$

and, if  $I - K_{z_1}$  and  $I - K_{z_2}$  are both Fredholm, then their indices are the same.

*Proof.* Let  $\tilde{z} := z_1 - z_2$ . We point out first that

$$\phi_n \xrightarrow{s} \phi \Rightarrow \tilde{z}\phi_n \rightarrow \tilde{z}\phi \Rightarrow K_{\tilde{z}}\phi_n \rightarrow K_{\tilde{z}}\phi,$$

so that, by Corollary 2.3,

$$\phi_n \xrightarrow{w^*} \phi \Rightarrow K_{\tilde{z}}K_{z_1}\phi_n \rightarrow K_{\tilde{z}}K_{z_1}\phi.$$

But, since every closed ball in  $X$  is weak\* sequentially compact, this result implies that  $K_{\bar{z}}K_{z_1}$  is a compact operator on  $Y$ . Clearly, by the same argument,  $K_{\bar{z}}^2$  is also compact, so that  $I + K_{\bar{z}}$  is Fredholm, having  $I - K_{\bar{z}}$  as a left and right regulariser. Since

$$(I + K_{\bar{z}})(I - K_{z_1}) = I - K_{z_2} - K_{\bar{z}}K_{z_1}, \tag{30}$$

it follows from standard properties of Fredholm operators [17] that  $I - K_{z_1}$  Fredholm  $\Rightarrow I - K_{z_2} - K_{\bar{z}}K_{z_1}$  Fredholm  $\Rightarrow I - K_{z_2}$  Fredholm. The same argument establishes the reverse implication.

Since  $I + tK_{\bar{z}}$  is Fredholm for  $0 \leq t \leq 1$  it has the same index (zero) throughout this range. It follows from (30) that if  $I - K_{z_1}$  and  $I - K_{z_2}$  are both Fredholm, then  $\text{ind}(I - K_{z_1}) = \text{ind}(I - K_{z_2} - K_{\bar{z}}K_{z_1}) = \text{ind}(I - K_{z_2})$ . ■

**COROLLARY 2.12.** *If  $k$  satisfies (A1)–(A3) and, for some  $\lambda \in \mathbf{C}$ ,  $\text{ess.sup}_{|x| \geq A} |z(x) - \lambda| \rightarrow 0$  as  $A \rightarrow \infty$ , then*

$$I - \lambda K, I - K_z \text{ injective} \Rightarrow I - K_z \text{ surjective}, \quad (I - K_z)^{-1} \in B(Y).$$

*Proof.* By Theorem 2.10,  $(I - \lambda K)^{-1} \in B(Y)$ , so that  $I - \lambda K$  is Fredholm of index zero. It follows from Theorem 2.11 that  $I - K_z$  is Fredholm of index zero and the result follows. ■

We are now almost in a position to prove Theorem 2.14, announced in Section 1. We need one further lemma (note that  $L^Q$  is defined in (14)).

**LEMMA 2.13.**  *$L^Q$  is weak\* sequentially compact iff  $Q$  is compact and convex.*

*Proof.* If  $L^Q$  is weak\* sequentially compact it is clear, from (26), that  $Q$  must be closed and bounded.  $Q$  must also be convex, for if  $q_1, q_2 \in Q$  and, for some  $t \in (0, 1)$ ,  $q := (1 - t)q_1 + tq_2 \notin Q$ , then, defining  $f \in L_\infty(\mathbf{R})$  by

$$f(s) = \begin{cases} q_1, & 0 \leq s < 1 - t, \\ q_2, & 1 - t \leq s < 1, \end{cases}$$

and by  $f(1 + s) = f(s)$ ,  $s \in \mathbf{R}$ , and defining  $z_m \in L^Q$ , for  $m \in \mathbf{N}$ , by  $z_m(x) = f(mx_1)$ ,  $x = (x_1, \dots, x_n) \in \Omega$ , it follows that  $z_m \xrightarrow{w^*} q \notin L^Q$  as  $m \rightarrow \infty$ , so that  $L^Q$  is not weak\* sequentially compact.

Conversely, if  $Q$  is compact and convex then, clearly,  $W = L^Q$  is convex and is bounded (in  $X$ ). Further, defining  $W^{(A)}$  and  $\Omega_A$  as after Eq. (26), it is an elementary exercise to show that  $W^{(A)}$  is closed in  $L_2(\Omega_A)$ , for all  $A > 0$ , and it follows that  $L^Q$  is weak\* sequentially compact from (28). ■

We are now ready to prove Theorem 2.14:

*Proof of Theorem 2.14.* Suppose that the conditions of Theorem 2.14 are satisfied. Then, using Lemma 2.13, it is easy to see that all the conditions of the first part of Theorem 2.8 are satisfied by  $W = L^Q$ . Further, given  $z \in L^Q$ , we can choose any  $q \in Q$  and define a sequence  $(z_n) \subset L^Q$  by

$$z_n(x) = \begin{cases} z(x), & |x| \leq n, \\ q, & |x| > n. \end{cases}$$

Then  $z_n \xrightarrow{w^*} z$  and, by Corollary 2.12,  $(z_n)$  satisfies condition (13). Thus Theorem 2.8 applies and we obtain Theorem 2.14. ■

*Proof of Corollary 2.15.* We first show that the spectrum of  $K$  contains 0. Let  $(\phi_n) \subset Y$  be any sequence such that  $\|\phi_n\| = 1$  for each  $n$  and, for every sequence  $(a_n) \subset S_\tau$ ,  $T_{a_n}\phi_n \xrightarrow{w^*} 0$ ; one choice is  $\phi_n(x) = \exp(-|x|^2)\cos(nx_1)$ . Then, by Corollary 2.3 and (9), it follows that, for every  $(a_n) \subset S_\tau$ ,  $T_{a_n}K\phi_n \xrightarrow{s} 0$ , from which and (10) it follows that  $K\phi_n \rightarrow 0$ . Thus the spectrum of  $K$  contains 0. The corollary now follows immediately from Theorem 2.10. ■

### 3. WIENER-HOPF EQUATIONS

We study in this section the Wiener-Hopf integral equation of the form (6) in which  $G$  is some measurable subset of  $\mathbf{R}^n$  ( $n \geq 2$ ), the functions  $\kappa \in L_1(\mathbf{R}^n)$  and  $\phi \in Y = BC(\mathbf{R}^n)$  are known, and  $\psi \in Y$  is to be determined.

For  $x \in \mathbf{R}^n$  and  $r > 0$  let  $B_r(x) = \{y \in \mathbf{R}^n: |x - y| < r\}$ . For  $r > 0$ ,  $0 < \alpha \leq \pi$ , and  $x, d \in \mathbf{R}^n$  with  $|d| = 1$ , denote by  $C(x, d, \alpha, r)$  the finite circular cone of interior angle  $\alpha$ ,

$$C(x, d, \alpha, r) := B_r(x) \cap \{y \in \mathbf{R}^n: (y - x) \cdot d > |y - x|\cos(\alpha/2)\}.$$

For  $r > 2/\pi$ , define  $U_r$  by

$$U_r := \{\Omega \subset \mathbf{R}^n: \Omega \text{ is open and } \forall x \in \partial\Omega \exists d \in \mathbf{R}^n \text{ with } |d| = 1 \text{ such that}$$

$$C(x, d, \alpha, r) \subset \Omega, C(x, -d, \alpha, r) \subset \mathbf{R}^n \setminus \bar{\Omega}, \text{ with } \alpha = \pi - 2/r\}.$$

For  $G \subset \mathbf{R}^n$  and  $r > 0$ , let  $rG = \{rx: x \in G\}$ . Then the following lemma implies that  $U_r$  is not empty.

LEMMA 3.1. *Let  $G \subset \mathbf{R}^n$  be a bounded open set of class  $C^1$ . Then, for any  $r > 2/\pi$ ,  $sG \in U_r$  for all sufficiently large  $s$ .*

*Proof.* Since  $G$  is a bounded open set of class  $C^1$ , there are open sets  $\Omega_j$  ( $j = 1, \dots, J$ ) in  $\mathbf{R}^n$  with  $\partial G \subset \cup_{j=1}^J \Omega_j$ , and corresponding  $C^1$  diffeomorphisms  $\phi_j$  ( $j = 1, \dots, J$ ) of  $\Omega_j$  onto an open ball of  $\mathbf{R}^n$  such that  $\phi_j(\Omega_j \cap G)$  is the part of this ball to one side of a hyperplane through its centre, for  $j = 1, \dots, J$ . Without loss of generality, we can assume that  $\phi_j$  maps  $\Omega_j$  to an open neighborhood of the origin in  $\mathbf{R}^n$  such that  $\phi_j(\Omega_j \cap G)$  consists of those points of  $\phi_j(\Omega_j)$  whose  $i$ th coordinate  $\phi_j^i$  is positive. For  $x \in \Omega_j \cap \partial G$ ,  $y \in \Omega_j \cap G$  we have, since  $\phi_j^i(x) = 0$ ,

$$\phi_j^i(y) = (y - x) \cdot \nabla \phi_j^i(x) + o(|y - x|)$$

as  $y \rightarrow x$ , and this holds uniformly in  $x$  on compact subsets of  $\Omega_j \cap \partial G$ . It follows that, given  $\epsilon > 0$ , there is, for  $j = 1, \dots, J$ , an  $R_j > 0$  and a compact subset  $S_j \subset \Omega_j \cap \partial G$  such that, for all  $x \in S_j$ ,

$$G \cap B_{R_j}(x) \subset \{y: (y - x) \cdot \nu(x) > -\epsilon|y - x|\}, \quad (31)$$

$$B_{R_j}(x) \cap \{y: (y - x) \cdot \nu(x) > \epsilon|y - x|\} \subset G, \quad (32)$$

where  $\nu(x) = \nabla \phi_j^i(x)/|\nabla \phi_j^i(x)|$  denotes the inner unit normal at  $x \in \partial G$ . Further, we can choose the compact subsets  $S_j$  so that  $\partial G \subset \cup_{j=1}^J S_j$ . By taking  $\epsilon = \cos(\alpha/2)$  and letting  $R$  be the smallest of  $R_1, \dots, R_J$ , it is seen from (31) and (32) that, for all  $s > 0$  and  $x \in \partial G$ ,

$$sG \cap B_{sR}(sx) \subset \{y: (y - sx) \cdot \nu(x) > -\epsilon|y - sx|\}, \quad (33)$$

$$B_{sR}(sx) \cap \{y: (y - sx) \cdot \nu(x) > \epsilon|y - sx|\} \subset sG. \quad (34)$$

Therefore, for  $x \in \partial(sG)$  with  $s \geq r/R$  it follows, on taking  $d = \nu(x/s)$ , that  $C(x, d, \alpha, r) \subset sG$  by (34) and  $C(x, -d, \alpha, r) \subset \mathbf{R}^n \setminus \overline{sG}$  by (33). This implies that  $sG \subset U_r$  for all  $s \geq r/R$ . ■

The following, an application of Theorem 2.4, is our first main result on the solvability of the Wiener-Hopf integral equation (6). Recall that  $\chi_G$  denotes the characteristic function of the set  $G$ .

THEOREM 3.2. *If  $\Omega = \mathbf{R}^n$ ,  $k(x, y) = \kappa(x - y)$ , with  $\kappa \in L_1(\mathbf{R}^n)$ , and  $1 - \hat{\kappa}(\xi) \neq 0$ ,  $\xi \in \mathbf{R}^n$ , then, for some  $r > 2/\pi$  and all  $G \subset U_r$ ,  $I - K_{\chi_G}$  is injective so that  $(I - K_{\chi_G})^{-1}$  exists as an operator on the range space  $(I - K_{\chi_G})Y$ . Moreover*

$$\sup_{G \in U_r} \|(I - K_{\chi_G})^{-1}\| < \infty. \quad (35)$$

Further, if  $G \subset U_r$  is bounded then  $I - K_{\chi_G}$  is surjective so that  $(I - K_{\chi_G})^{-1} \in B(Y)$ .

*Proof.* As stated in the Introduction, with this choice of  $\Omega$  and  $k$ , assumptions (A1)–(A3) are satisfied with  $\omega = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n: 0 \leq x_j < 1, j = 1, \dots, n\}$  and  $\tau = \{e_1, \dots, e_n\}$ , so that condition (i) of Theorem 2.4 is satisfied.

Let  $W = \{1, 0, \chi_G: G \text{ is a half-space}\}$  and, for  $n \in \mathbf{N}$ , let  $W_n = \{\chi_G: G \in U_n\}$ . Then it is easy to see that condition (ii) of Theorem 2.4 is satisfied and, since  $1 - \hat{\kappa}(\xi) \neq 0, \xi \in \mathbf{R}^n$ , condition (iv) is also satisfied: see [14, 13]. It remains to show that condition (iii) of Theorem 2.4 is satisfied.

Let  $(z_n) \in W^\times$ . Then, for each  $n, z_n = \chi_{G_n}$  with  $G_n \in U_n$ . There are two cases to be distinguished.

*Case 1.* For all  $A > 0, B_A(0) \cap \partial G_{W_n^*} = \emptyset$  for all sufficiently large  $n$ .

In this case, it is easy to see that  $z_n \rightarrow 0$  or  $1$  as  $n \rightarrow \infty$ .

*Case 2.* There exists a bounded sequence  $(x_{n_j})$  such that  $x_{n_j} \in \partial G_{n_j}$  for each  $j$ .

In this case, there is a subsequence of  $(x_{n_j})$ , which is simply denoted by itself, such that  $x_{n_j} \rightarrow x$  as  $j \rightarrow \infty$  for some  $x \in \mathbf{R}^n$ . Let  $d_{n_j} \in \mathbf{R}^n$  be such that

$$\begin{aligned} |d_{n_j}| = 1, \quad C(x_{n_j}, d_{n_j}, \alpha_j, n_j) \subset G_{n_j}, \\ C(x_{n_j}, -d_{n_j}, \alpha_j, n_j) \subset \mathbf{R}^n \setminus \bar{G}_{n_j}, \end{aligned} \tag{36}$$

with  $\alpha_j = \pi - 2/n_j$ . Then there is a subsequence of  $(d_{n_j})$ , denoted simply by itself, such that  $d_{n_j} \rightarrow d$  as  $j \rightarrow \infty$  for some  $d \in \mathbf{R}^n$  with  $|d| = 1$ . For  $x, d \in \mathbf{R}^n$ , let  $H(x, d)$  denote the halfplane  $H(x, d) = \{y | (y - x) \cdot d > 0\}$  and let  $m(G)$  denote the measure of a measurable set  $G$ . Then it is easy to see that, for all  $A > 0$ ,

$$\begin{aligned} m\left(\left(H(x, d) \setminus G_{n_j}\right) \cap B_A(0)\right) \\ \leq m\left(\left(H(x, d) \setminus C(x_{n_j}, d_{n_j}, \alpha_j, n_j)\right) \cap B_A(0)\right) \rightarrow 0, \\ m\left(\left(G_{n_j} \setminus H(x, d)\right) \cap B_A(0)\right) \\ \leq m\left(\left(H(x, -d) \setminus C(x_{n_j}, -d_{n_j}, \alpha_j, n_j)\right) \cap B_A(0)\right) \rightarrow 0 \end{aligned}$$

as  $j \rightarrow \infty$ . Therefore  $z_{n_j} \xrightarrow{w^*} \chi_{H(x, d)}$  as  $j \rightarrow \infty$  so that (iii) is verified and Theorem 2.4 can be applied to obtain the results of the theorem up to Eq. (35). The last statement of the theorem follows from Corollary 2.12. ■

The following corollary, originally established by Widom [21], follows immediately from Lemma 3.1 and Theorem 3.2:

**COROLLARY 3.3.** *If the conditions of Theorem 3.2 are satisfied and  $G$  is a bounded open set of class  $C^1$  then, for some  $r_0 > 0$  and all  $r \geq r_0$ ,  $(I - K_{z_r})^{-1} \in B(Y)$ , where  $z_r := \chi_{rG}$ , and  $\sup_{r \geq r_0} \|(I - K_{z_r})^{-1}\| < \infty$ .*

Now we consider the case when  $G$  is a quadrant. In this case, we prove, by using Theorem 3.2, the known result that  $I - K_{\chi_G}$  is Fredholm of index zero provided  $1 - \hat{\kappa}(\xi) \neq 0$ ,  $\xi \in \mathbf{R}^2$ . To this end, we establish first the following result.

**LEMMA 3.4.** *Let  $G \subset \mathbf{R}^2$  be a quadrant. Then, for any  $t > 0$ , there exists a sequence of bounded open sets  $(G_n) \subset U_t$  such that  $G_{n-1} \subset G_n \subset G$  and  $G_n \setminus G_{n-1} \subset G \setminus B_{n-1}(0)$  for  $n = 2, 3, \dots$  and the set  $G \setminus (\bigcup_{n=1}^{\infty} G_n)$  is bounded.*

*Proof.* For  $r_2 > 2r_1 > 0$  it is easy to see that there exists a bounded open set  $Q \subset G$  of class  $C^1$  which coincides with  $G$  in the annulus  $B_{r_2}(0) \setminus B_{r_1}(0)$ . By Lemma 3.1 we obtain that for any  $t > 0$  there is a number  $T > 0$  such that  $TQ \in U_t$ . Then, letting  $G_1 = TQ$  and setting  $G_n = G_{n-1} \cup (2G_{n-1})$  for  $n = 2, 3, \dots$ , it is easy to see that the sequence  $(G_n)$  has the desired properties. ■

With the help of Lemma 3.4 and Theorem 3.2 the following result can be easily proved.

**THEOREM 3.5.** *Suppose that  $\Omega = \mathbf{R}^2$ ,  $k(x, y) = \kappa(x - y)$ , with  $\kappa \in L_1(\mathbf{R}^n)$ , and let  $G \subset \mathbf{R}^2$  be a quadrant. If  $1 - \hat{\kappa}(\xi) \neq 0$  for all  $\xi \in \mathbf{R}^2$ , then  $I - K_{\chi_G}$  is Fredholm of index zero on  $Y$ .*

*Proof.* Choose  $t$  sufficiently large so that, by Theorem 3.2,  $(I - K_{\chi_S})^{-1}$  exists and is uniformly bounded in  $S$  for  $S \in U_t$ . Now choose a sequence of domains  $(G_n) \subset U_t$  as indicated in Lemma 3.4 and let  $G_\infty = \bigcup_{n=1}^{\infty} G_n$  and  $z_n = \chi_{G_n}$ ,  $z_\infty = \chi_{G_\infty}$ . Then, for  $n = 1, 2, \dots$ , the bounded set  $G_n \in U_t$ , so that  $(I - K_{z_n})^{-1} \in B(Y)$  exists and is uniformly bounded in  $n$ , and  $G_\infty \in U_t$  so that  $I - K_{z_\infty}$  is injective. Obviously  $z_n \xrightarrow{w^*} z_\infty$  as  $n \rightarrow \infty$  so that, by Theorem 2.6,  $(I - K_{z_\infty})^{-1} \in B(Y)$ . Thus  $I - K_{z_\infty}$  is Fredholm of index zero. As the difference between  $G$  and  $G_\infty$  is bounded,  $I - K_{\chi_G}$  is Fredholm of index zero by Theorem 2.11. ■

Simonenko [19], in the function space setting  $Y = L_p(\Omega)$ ,  $1 \leq p < \infty$ , has shown the same result that  $I - K_{\chi_G}$  is Fredholm of index zero provided  $1 - \hat{\kappa}(\xi) \neq 0$ ,  $\xi \in \mathbf{R}^n$ , for the more general case when  $G$  is a smooth cone in  $\mathbf{R}^n$ ; that is,  $rG = G$  for  $r > 0$  and the boundary of  $G$  is a  $C^1$  surface except at the origin. (See Böttcher *et al.* [3] for the same result in the case  $p = \infty$ .) Our arguments apply to more general cones  $G$  of this

type if a similar result to Lemma 3.4 holds; specifically if, for some  $r_2 > 2r_1 > 0$ , there exists a bounded open set  $Q \subset G$  of class  $C^1$  which coincides with  $G$  in the annulus  $B_{r_2}(0) \setminus B_{r_1}(0)$ .

#### 4. THE LIPPMANN-SCHWINGER EQUATION

As indicated in Section 1 we consider now the Lippmann-Schwinger integral equation (15), with  $\lambda \in X$ , considering first the case in which  $\lambda(x) = \lambda_0$  outside a bounded open set  $\Omega$  for some  $\lambda_0 \in D := \{w: \Re w > 0, \Im w \geq 0\}$ . We suppose that  $\lambda \in L^\Lambda$  (defined by (14)), with  $\Lambda$  a compact subset of  $D$ . In this case of  $\lambda_0^2 - \lambda^2$  having compact support it is well known (see, for example, [10, Theorem 8.7]) that, provided  $\phi := u^i|_{\bar{\Omega}} \in BC(\bar{\Omega})$ , Eq. (15) has exactly one solution  $\psi := u|_{\bar{\Omega}} \in BC(\bar{\Omega})$ , and  $\psi$  depends continuously on  $\phi$ .

Defining additionally

$$k(x, y) = \Phi_{\lambda_0}(x - y), \quad z(y) = \lambda_0^2 - \lambda^2(y), \quad x, y \in \bar{\Omega}, \quad (37)$$

the Lippmann-Schwinger equation can be written as Eq. (1). As  $\Phi_{\lambda_0} \in L_1^{loc}(\mathbf{R}^n)$ , (A1) and (A2) are satisfied by  $k$  and, as  $\Omega$  is bounded, (A3) is trivially satisfied. Let  $Q$  denote the convex hull of  $\{\lambda_0^2 - w^2: w \in \Lambda\}$ , so that  $Q = \{\lambda_0^2 - w^2: w \in \Lambda_H\}$  for some compact subset  $\Lambda_H$  of  $D$ . Then  $Q$  is compact and convex and, defining  $K_z$  by (2) and (4),  $I - K_z$  is injective for every  $z \in L^Q$  (since (15) is uniquely solvable for  $\lambda \in L^{\Lambda_H}$ ) so that the conditions of Theorem 2.14 are satisfied and Theorem 2.14 can be applied to obtain the stability bound (16).

We consider now the case when  $\lambda$  is not compactly supported and have in mind the particular case when the incident field is a plane wave travelling in the  $x_1$  direction so that

$$u^i(x) = e^{i\lambda_0 x_1}, \quad x \in \mathbf{R}^n. \quad (38)$$

In this case we redefine  $\Omega = \mathbf{R}^n$  and, since  $\Phi_{\lambda_0} \notin L_1(\mathbf{R}^n)$  unless  $\Im \lambda_0 > 0$ , we restrict attention to the case when  $\Re \lambda_0 > 0$ ,  $\Im \lambda_0 > 0$ . Since, with  $\Im \lambda_0 > 0$ ,  $u^i(x)$  increases exponentially in the negative  $x_1$  direction, we consider the solvability of (15) in spaces of bounded continuous functions with exponential weights. For  $\eta \in \mathbf{R}^n$  we define the weighted space  $Y(\eta)$  by (17) and note that, with this definition,  $u^i \in Y(-\Im \lambda_0 e_1)$ , where  $e_1$  is the unit vector in the  $x_1$  direction. Rewriting (15) as an equation for the scattered field, namely (18), with  $F$  given by (19), and realising that  $F$  will not be bounded in general if  $u^i$  is unbounded and  $\lambda_0^2 - \lambda^2$  is not

compactly supported, we introduce new weighted functions to which we will apply the theory of Section 2. Precisely, fixing  $\eta \in \mathbf{R}^n$  with  $|\eta| < \Im \lambda_0$ , define  $\phi$ ,  $\psi$ , and  $k$  by

$$\begin{aligned}\phi(x) &= F(x)e^{-x \cdot \eta}, & \psi(x) &= u^s(x)e^{-x \cdot \eta}, & x &\in \mathbf{R}^n, \\ k(x, y) &= \Phi_{\lambda_0}(x - y)e^{(x-y) \cdot \eta}, & x, y &\in \mathbf{R}^n, x \neq y, & & (39)\end{aligned}$$

and  $z \in X$  by (37). Then Eq. (18) can be written as (1). Since  $\Phi_{\lambda_0}(x) = O(e^{-\Im \lambda_0 |x|})$  as  $x \rightarrow \infty$  and  $|\eta| < \Im \lambda_0$ , the function  $\Phi_{\lambda_0}(x)e^{x \cdot \eta} \in L_1(\mathbf{R}^n)$  and, as discussed after Eq. (8), assumptions (A1)–(A3) are satisfied by  $k$  and  $\Omega$ . Since, with  $\phi$ ,  $k$ , and  $\psi$  thus defined, (18) has a solution in  $Y(\eta)$  iff (1) has a solution in  $Y$ , we obtain, applying Theorem 2.14, the following solvability result.

**THEOREM 4.1.** *Suppose that  $H$  is some compact convex subset of  $\{re^{i\theta} : r > 0, 0 < \theta < 2\pi\}$  and that, for some  $\eta \in \mathbf{R}^n$ , the homogeneous version of (18) has only the trivial solution in  $Y(\eta)$  for all  $\lambda^2 \in L^H$ . Then, if  $\lambda^2 \in L^H$  and  $F \in Y(\eta)$ , Eq. (18) has exactly one solution  $u^s \in Y(\eta)$  and*

$$\|u^s\|_\eta \leq C(\eta, H)\|F\|_\eta.$$

To utilise this theorem we prove the following uniqueness result.

**THEOREM 4.2.** *If for some  $b > a > 0$ ,  $0 < \theta < \pi$ ,  $\lambda \in L^\Lambda$  with  $\Lambda = \{re^{i\theta} : a \leq r \leq b\}$  and  $|\eta| < \min_{z \in \Lambda} \Im z = a \sin \theta$ , then the homogeneous version of (18) has only the trivial solution in  $Y(\eta)$ .*

*Proof.* Choose  $\epsilon$  in the range  $|\eta| < \epsilon < a \sin \theta$  and define  $\Psi \in Y$  by  $\Psi(x) = \exp(-2\epsilon\sqrt{1 + |x|^2})$ ,  $x \in \mathbf{R}^n$ .

Suppose that  $f \in Y(\eta)$  and define the volume potential  $v$  by

$$v(x) = \int_{\mathbf{R}^n} \Phi_{\lambda_0}(x - y)f(y) dy, \quad x \in \mathbf{R}^n. \quad (40)$$

Then

$$e^{-x \cdot \eta} v(x) = \int_{\mathbf{R}^n} k(x, y)e^{-y \cdot \eta} f(y) dy, \quad x \in \mathbf{R}^n,$$

with  $k$  as defined above, so that  $v \in Y(\eta)$  and  $\|v\|_\eta \leq \|K\| \|f\|_\eta$ , where  $K$  is the integral operator defined by (2). Further, it is not difficult to see, using standard mapping properties (e.g., [10, Theorem 8.2]) that  $v \in H_{loc}^2(\mathbf{R}^n) \cap C^1(\mathbf{R}^n)$  and, in a distributional sense,

$$\Delta v + \lambda_0^2 v = -f.$$

Further, from local regularity results [12], it follows that  $\nabla v \in Y(\eta)$ .

It follows that if  $u \in Y(\eta)$  satisfies the homogeneous version of (18) then  $\Delta u + \lambda^2 u = 0$  in  $\mathbf{R}^n$  and that, integrating by parts,

$$\begin{aligned} 0 &= e^{-i\theta} \int_{\mathbf{R}^n} (\Delta u + \lambda^2 u) \bar{u} \Psi \, dx \\ &= \int_{\mathbf{R}^n} (\lambda|\lambda| |u|^2 - e^{-i\theta} |\nabla u|^2) \Psi \, dx - e^{-i\theta} \int_{\mathbf{R}^n} \bar{u} \nabla u \cdot \nabla \Psi \, dx. \end{aligned}$$

Taking the imaginary part of this equation, noting that  $|\nabla \Psi| \leq 2\epsilon |\Psi|$  and using the Cauchy-Schwarz inequality, we obtain that

$$(p^2 + q^2) \sin \theta \leq \Im \int_{\mathbf{R}^n} (\lambda|\lambda| |u|^2 - e^{-i\theta} |\nabla u|^2) \Psi \, dx \leq \frac{2\epsilon}{a} pq,$$

where

$$p^2 = a^2 \int_{\mathbf{R}^n} |u|^2 \Psi \, dx, \quad q^2 = \int_{\mathbf{R}^n} |\nabla u|^2 \Psi \, dx.$$

Since  $\epsilon < a \sin \theta$  it follows that  $p = q = 0$  and  $u \equiv 0$ . ■

Combining Theorems 4.1 and 4.2 we obtain the following corollary.

**COROLLARY 4.3.** *If for some  $b > a > 0$ ,  $0 < \theta < \pi$ ,  $\lambda \in L^\Lambda$  with  $\Lambda = \{re^{i\theta} : a \leq r \leq b\}$ , and  $F \in Y(\eta)$  with  $|\eta| < \min_{z \in \Lambda} \Im z = a \sin \theta$ , then Eq. (18) has exactly one solution  $u^s \in Y(\eta)$  and*

$$\|u^s\|_\eta \leq C(\eta, \Lambda) \|F\|_\eta.$$

We conjecture that Theorem 4.2 holds with  $\Lambda$  an arbitrary compact subset of the upper half-plane, but have been unable to modify the proof to deal with this case.

The condition on  $F$  in Corollary 4.3 holds in many cases. In particular, if  $u^i$  is the plane wave (38) and  $\lambda = \lambda_0$  in the half-plane  $x_1 < \alpha x_2$ , for some  $\alpha \in \mathbf{R}$ , then  $F$  is a volume potential of the form (40) with density  $f \in Y(\eta)$ , with  $\eta = -\alpha \Im \lambda_0 u / \sqrt{1 + \alpha^2}$ , where  $u$  is the unit vector  $u = (\alpha e_1 + e_2) / \sqrt{1 + \alpha^2}$ . It follows from the mapping properties in the proof of Theorem 4.2 that also  $F \in Y(\eta)$ , and  $|\eta| < \epsilon := \min_{z \in \Lambda} \Im z$  if  $|\alpha| \sqrt{(\Im \lambda_0)^2 - \epsilon^2} < \epsilon$ , in particular if  $\epsilon = \Im \lambda_0$ .

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