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Numerical analysis of graded mesh methods for a class of second kind integral equations on the real line

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Abstract

In this paper we are concerned with the numerical analysis of the collocation method based on graded meshes of second kind integral equations on the real line of the form

$$\phi(s) = \psi(s) + \int_{\mathbb{R}} \kappa(s-t)z(t)\phi(t)\,dt, \quad s \in \mathbb{R},$$

where $\kappa \in L^1(\mathbb{R})$, $z \in L^\infty(\mathbb{R})$, and $\psi \in BC(\mathbb{R})$, the space of bounded continuous complex-valued functions on $\mathbb{R}$, are assumed known and the function $\phi \in BC(\mathbb{R})$ is to be determined. We introduce some new graded meshes for the collocation method of the integral equation, which are different from those used previously for the Wiener–Hopf integral equation in the case when the solution decays exponentially at infinity, and establish optimal local and global L^∞-norm error estimates under the condition that the solution decays only polynomially at infinity.
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1. Introduction

This paper is concerned with the numerical analysis of graded mesh methods for second kind integral equations on the real line of the form

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$$\phi(s) = \psi(s) + \int_{\mathbb{R}} \kappa(s-t)z(t)\phi(t)\,dt, \quad s \in \mathbb{R}, \tag{1.1}$$

where $\kappa \in L^1(\mathbb{R})$, $z \in L^\infty(\mathbb{R})$, and $\psi \in BC(\mathbb{R})$, the space of bounded continuous complex-valued functions on $\mathbb{R}$, are assumed known. (We assume throughout that $\kappa \not\equiv 0$.) The function $\phi \in BC(\mathbb{R})$ is to be determined.

Equation (1.1) can be abbreviated in operator form as

$$\phi = \psi + K(z\phi) = \psi + K_z\phi, \tag{1.2}$$

where the integral operator $K : L^\infty(\mathbb{R}) \to BC(\mathbb{R})$ is defined by

$$K\phi(s) = \int_{\mathbb{R}} \kappa(s-t)\phi(t)\,dt, \quad s \in \mathbb{R}, \tag{1.3}$$

and, for $z \in L^\infty(\mathbb{R})$, $K_z : L^\infty(\mathbb{R}) \to BC(\mathbb{R})$ is defined by $K_z\phi = K(z\phi)$ for $\phi \in L^\infty(\mathbb{R})$. Throughout this paper we will assume that (1.1) is uniquely solvable in $BC(\mathbb{R})$ for every $\psi \in BC(\mathbb{R})$, so that $(I - K_z)^{-1} : BC(\mathbb{R}) \to BC(\mathbb{R})$ exists and is bounded. In fact, the unique solvability of (1.1) has been studied previously in [12,16,17]. Let $\|L\|$ denote the norm of a bounded operator $L : BC(\mathbb{R}) \to BC(\mathbb{R})$ and, for some $Q \subset \mathbb{C}$,

$$L^Q := \bigl\{ z \in L^\infty(\mathbb{R}) \bigm| z(s) \in Q \text{ for almost all } s \in \mathbb{R} \bigr\}.$$

Using this notation, the following theorem has been obtained [16].

Theorem 1.1. *If $Q \subset \mathbb{C}$ is compact and convex and if $I - K_z : BC(\mathbb{R}) \to BC(\mathbb{R})$ is injective for all $z \in L^Q$, then $I - K_z$ is bijective for all $z \in L^Q$ and $\sup_{z \in L^Q} \|(I - K_z)^{-1}\| < \infty$.*

The numerical method proposed will be based on the finite section approximation of (1.1) by

$$\phi_A(s) = \psi(s) + \int_{-A}^{A} \kappa(s-t)z(t)\phi_A(t)\,dt, \quad s \in \mathbb{R} \tag{1.4}$$

for some $A > 0$, which can be rewritten in operator form as

$$\phi_A = \psi + K_{z,A}\phi_A, \tag{1.5}$$

where the operator $K_{z,A}$ is defined by

$$K_{z,A}\psi(s) = \int_{-A}^{A} \kappa(s-t)z(t)\psi(t)\,dt, \quad s \in \mathbb{R}. \tag{1.6}$$

The following theorem, which ensures the stability of the finite section approximation (1.4), has been obtained in [12, Theorem 4.5].

Theorem 1.2. *If $Q \subset \mathbb{C}$ is compact and convex and if $I - K_z : BC(\mathbb{R}) \to BC(\mathbb{R})$ is injective for all $z \in L^Q$, then, for some $A_0 \geqslant 0$, $I - K_{z,A} : BC(\mathbb{R}) \to BC(\mathbb{R})$ is bijective for all $A \geqslant A_0$ and $z \in L^Q$, with*

$$C_1 := \sup_{z \in L^Q,\, A \geqslant A_0} \bigl\|(I - K_{z,A})^{-1}\bigr\| < \infty. \tag{1.7}$$

The convergence analysis of the finite section approximation and the graded mesh methods will depend crucially on the asymptotic behaviour at infinity of solutions of Eq. (1.1). For $a \geqslant 0$, let X_a denote the weighted space of continuous functions defined by

$$X_a := \{\phi \in BC(\mathbb{R}) \mid \phi(s) = O(|s|^{-a}),\ |s| \to \infty\}$$

with the norm $\|\cdot\|_{X_a}$, defined by $\|\phi\|_{X_a} = \|\phi w_a\|_{L^\infty}$, where $w_a(s) = (1+|s|)^a$, and denote by $\|A\|_{X_a}$ the norm of a bounded operator $A : X_a \to X_a$. Then the following result, generalising the results of [11–13,15], has been established in [4].

Theorem 1.3. *Suppose $\kappa \in L^1$ and $\kappa(s) = O(|s|^{-b})$ as $|s| \to \infty$, for some $b > 1$, $Q \subset \mathbb{C}$ is compact and convex. Then $(I - K_z)^{-1} : X_a \to X_a$ exists and is bounded for all $z \in L^Q$ with*

$$\sup_{z\in L^Q} \|(I - K_z)^{-1}\|_{X_a} < \infty$$

if and only if $I - K_z : BC(\mathbb{R}) \to BC(\mathbb{R})$ is injective for all $z \in L^Q$.

The integral equation of the form (1.1) arises in the study of acoustic or electromagnetic scattering by an impedance half-plane using the integral equation method (see, e.g., [12, 14,29]). For the integral equation obtained in [14,29] for the problem of scattering by an impedance half-plane, Theorem 1.3 can be applied with $b = 3/2$ under certain conditions on the surface impedance [4].

In the special case when $z = \chi_{(0,\infty)}$ is the characteristic function of the half-line $(0,\infty)$, (1.1) becomes the Wiener–Hopf integral equation

$$\phi(s) = \psi(s) + \int_0^\infty \kappa(s-t)\phi(t)\,dt, \quad s \in \mathbb{R}, \tag{1.8}$$

the numerical treatment of which has been widely studied in the literature (see, e.g., [1–3, 9,19–21,24,26,27] and the references therein).

For the Wiener–Hopf case (1.8), as cited above, the literature is considerable and very suitable numerical schemes have been proposed, but invariably for the case when the solution ϕ decays exponentially at infinity. Then it is appropriate in a piecewise polynomial approximation to ϕ to use a graded mesh with the spacing between mesh points increasing with distance from the origin, and to aim to obtain, e.g., in the uniform norm on $[0,\infty)$, an optimal order of convergence as the number of degrees of freedom is increased [1,9,19,20, 24,26], even exponential convergence using $h - p$ methods [21].

However, as seen from Theorem 1.3, for problems where κ and ψ only decay polynomially at infinity (for example, the acoustic or electromagnetic scattering problems) the solution ϕ of the integral equation (1.1) or the Wiener–Hopf integral equation (1.8) also decays only polynomially at infinity. Thus it is very important to construct efficient and fast graded meshes for the numerical computation of solutions of integral equations on the real line with solutions decaying only polynomially at infinity and to establish optimal orders of convergence of the corresponding methods for integral equations on the real line. Our main concern in this paper is to solve the integral equation (1.1) including the Wiener–Hopf integral equation (1.8) in the case where both the kernel κ and the known function ψ

only exhibit polynomial decay at infinity. In Section 3 we will introduce some new graded meshes for the collocation method of (1.1) including the Wiener–Hopf case (1.8). These graded meshes are different from those used previously for the Wiener–Hopf integral equation (1.8) in the case when the solution decays exponentially at infinity. In Sections 4 and 5 we will establish optimal global L^∞-norm error estimates for the case when the solution decays only polynomially at infinity. For practical computation in obtaining the approximate solution in a finite interval, we introduce a new interval approximation scheme in Section 6, which is proved to decrease the grid points with increased accuracy. The optimal local L^∞-norm error estimate over the finite interval is also obtained again for the case when the solution decays only polynomially at infinity. It should be remarked that a fast two-grid piecewise constant collocation scheme on a uniform grid on $\mathbb{R}$ for solving (1.1) is proposed recently in [18] in the case where the solution does not exhibit decay at infinity.

2. Numerical schemes

To solve (1.1) numerically, we first approximate it by the finite section equation (1.4). Equation (1.4) is then discretised by a numerical method. In this paper we will consider the collocation method based on graded meshes.

Let Π_n denote the mesh partition of the real line $\mathbb{R}$ and let $S_{\pm i}^{(n)}$ $(i = 0, 1, \dots)$ be the nodes such that

$$-\infty < \cdots < S_{-i}^{(n)} < \cdots < S_{-1}^{(n)} < S_{-0}^{(n)} = 0 = S_0^{(n)} < S_1^{(n)} < \cdots < S_i^{(n)} < \cdots < +\infty,$$

where n is a positive integer. Set $I_i^{(n)} = [S_i^{(n)}, S_{i+1}^{(n)}]$, $I_{-i}^{(n)} = [S_{-(i+1)}^{(n)}, S_{-i}^{(n)}]$ and let $h_{\pm i}^{(n)} = |S_{\pm(i+1)}^{n} - S_{\pm i}^{(n)}|$ be the step length of the sub-intervals $I_{\pm i}^{(n)}$. Denote by $V_r(\Pi_n)$ the space of piecewise polynomials on Π_n with index $r \geqslant 1$ and with polynomials of degree not greater than $r-1$ on each sub-interval $I_{\pm i}^{(n)}$.

Let $\{\xi_j \mid 1 \leqslant j \leqslant r\}$ be the basic quadrature nodes with

$$0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_r \leqslant 1$$

and let $S_{\pm ij}^{(n)}$ be the nodes for the polynomial function of degree $r-1$ on each $I_{\pm i}^{(n)}$ with

$$S_{ij}^{(n)} = S_i^{(n)} + \xi_j h_i^{(n)}, \quad j = 1, 2, \dots, r,$$
$$S_{-ij}^{(n)} = S_{-(i+1)}^{(n)} + \xi_j h_{-i}^{(n)}, \quad j = 1, 2, \dots, r.$$

Let $P_n : BC(\mathbb{R}) \to V_r(\Pi_n)$ be the interpolation projection operator defined as follows:

$$(P_n v)(s) = \sum_{j=1}^{r} l_{\pm ij}^{(n)}(s) v\big(s_{\pm ij}^{(n)}\big), \quad s \in I_{\pm i}^{(n)},\ i = 0, 1, 2, \dots \tag{2.1}$$

for $v \in BC(\mathbb{R})$, where

$$l_{\pm ij}^{(n)}(s) = \prod_{k=1,\, k \neq j}^{r} \frac{\big(s - S_{\pm ik}^{(n)}\big)}{\big(S_{\pm ij}^{(n)} - S_{\pm ik}^{(n)}\big)}$$

are the Lagrangian basis functions. Then the discrete scheme for (1.4) can be defined as

$$\phi_n(s) = \psi(s) + \int_{-A_n}^{A_n} \kappa(s-t)z(t)P_n\phi_n(t)\,dt, \tag{2.2}$$

where $A_n > 0$ with $A_n \to +\infty$ as $n \to \infty$; it is defined and analyzed in the following sections. Equation (2.2) can be rewritten in the operator form

$$\phi_n = \psi + K_{z,A_n}P_n\phi_n. \tag{2.3}$$

Remark 2.1. Equation (2.2) will be solved first at the collocation nodes $S^{(n)}_{\pm ij}$ to get the values of $\phi_n(S^{(n)}_{\pm ij})$ for $i = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, r$. These values will then be substituted back in (2.2) to obtain the numerical solution $\phi_n(s)$ for $s \in \mathbb{R}$.

It is expected that if the appropriate knowledge of the asymptotic behavior at infinity of ϕ is available, then the following optimal error estimate holds:

$$\|\phi - \phi_n\|_{L^\infty(\mathbb{R})} = O\left(n^{-r}\right). \tag{2.4}$$

This is the optimal result for second-kind integral equations in the case with compact operators if $h = 1/n$ (see [5,8]). This result was also proved for the half-line case with graded meshes under the assumption that the solution $\phi(s)$ is exponentially decay at infinity (see, e.g., [1,9]). In this paper, we only assume that the solution $\phi(s)$ is polynomially decay at infinity:

$$\phi^{(l)}(s) \approx O\left(|s|^{-p}\right), \quad 0 \leqslant l \leqslant r,\ s \to \pm\infty$$

for some $p > 0$, where $\phi^{(l)}(s)$ denotes the lth order derivative of $\phi(s)$. It will be shown that the optimal error estimate (2.4) remains true with a standard or uniform mesh and some new graded meshes as defined in the next section.

We conclude this section with introducing some function spaces. For $p > 0$ and a non-negative integer r define

$$BC^r_p(\mathbb{R}) := \Big\{\psi \in C(\mathbb{R}) \;\Big|\; \psi^{(l)} \in C(\mathbb{R}),\ 0 \leqslant l \leqslant r,$$
$$\|\psi\|_{BC^r_p} := \max_{0\leqslant l\leqslant r} \left\|\omega_p\psi^{(l)}\right\|_{L^\infty} < \infty\Big\},$$
$$BC^r_{p+r}(\mathbb{R}) := \Big\{\psi \in C(\mathbb{R}) \;\Big|\; \psi^{(l)} \in C(\mathbb{R}),\ 0 \leqslant l \leqslant r,$$
$$\|\psi\|_{BC^r_{p+r}} := \max_{0\leqslant l\leqslant r} \left\|\omega_{p+r}\psi^{(l)}\right\|_{L^\infty} < \infty\Big\},$$

where $\omega_p(s) = (1+|s|)^p$ for $s \in \mathbb{R}$. We write $BC_p(\mathbb{R}) = BC^0_p(\mathbb{R})$.

3. Partition meshes and their properties

In this section, we introduce several partition meshes and discuss their properties. To this end let n be a positive integer.

(I) *The standard mesh* (*or uniform mesh*) Π_n^u: $h_{\pm i}^{(n)} = h = 1/n$ and

$$S_0^{(n)} = 0, \qquad S_{i+1}^{(n)} = S_i^{(n)} + h_i^{(n)}, \qquad S_{-(i+1)}^{(n)} = S_{-i}^{(0)} - h_{-i}^{(n)}, \quad i = 1, 2, \ldots.$$

(II) *The iterative graded mesh* Π_n^i: $h_0^{(n)} = h = 1/n$, $S_0^{(n)} = 0$, and

$$h_{\pm i}^{(n)} = h\big(1 + \big|S_{\pm i}^{(n)}\big|\big)^{q_{\pm i}}, \qquad S_{\pm(i+1)}^{(n)} = S_{\pm i}^{(n)} \pm h_{\pm i}^{(n)}, \quad i = 0, 1, \ldots,$$

where $q_{\pm i} > 0$ are given constants.

(III) *The exponentially graded mesh* Π_n^e:

$$S_i^{(n)} = e^{i/(\alpha n)} - 1, \qquad S_{-i}^{(n)} = -S_i^{(n)}, \quad i = 0, 1, \ldots,$$

where $\alpha > 0$ is a given constant.

(IV) *The polynomially graded mesh* Π_n^p:

$$S_i^{(n)} = \left(1 + \frac{i}{q\alpha n}\right)^q - 1, \qquad S_{-i}^{(n)} = -S_i^{(n)}, \quad i = 0, 1, \ldots,$$

where $q \geqslant 1$ and $\alpha > 0$ are given constants.

Remark 3.1. There is a very large literature on graded mesh methods and approximations (see, e.g., [1,6–10,19,20,22–26,28,30,31] and the references quoted there). In order to obtain an optimal piecewise polynomial approximation to s^γ, $\gamma > 0$, on [0, 1], Rice [28] first introduced the graded mesh

$$\xi_i = \left(\frac{i}{n}\right)^q, \quad i = 0, 1, \ldots, n, \tag{3.1}$$

where $q > 1$ is called the *grading exponent*. The underlying idea of these graded meshes is that as q increases, more mesh points are placed near 0 so functions with singularities at 0 can be better approximated with the appropriate grading exponent q. In fact, in [28] it is shown, for $0 < \gamma < 1$, that s^γ for $s \in [0, 1]$ is optimally approximated in L^2 norm, using piecewise polynomials of degree $\leqslant \nu$ on the graded mesh (3.1), by taking $q = (3+2\nu)/(1+2\gamma)$. This type of graded meshes was used to obtain optimal orders of convergence in [8,31] for product integration methods for weakly singular integral equations of the second kind with compact integral operators and in [10,22,25] (see also [8]) for collocation methods for a class of second-kind integral equations in which the integral operator is not compact. For the application of this type of graded meshes to boundary integral equations on domains with corners see, e.g., [6–8,10,23,25] and the references quoted there. For integral equations of the second kind on the half-line including the Wiener–Hopf equations optimal orders of convergence (similar to (2.4)) have been obtained in [1,9,19,20,24,26,30] for collocation and quadrature methods in the case when the solution is assumed to be exponential decay at infinity, by using the graded meshes of the following type:

$$s_i = \frac{r}{\mu} \ln\left(\frac{m}{m+1-i}\right), \quad i = 1, 2, \ldots, n, \tag{3.2}$$

where $m \geqslant n$. Such a mesh is referred to as a (r, μ)-graded mesh in [1,9,10,25]. The underlying idea of a (r, μ)-graded mesh is that as r/μ increases, more mesh points are placed further from $s = 0$, and solutions with slower decay at ∞ can be better approximated.

Remark 3.2.

(i) Our polynomially graded mesh Π_n^p has a similar order form with (3.1). However, the new polynomially graded mesh Π_n^p in this paper is designed for integral equations on the real line, in which the index i will reach $m(n) \approx q\alpha n^{1+r/(pq)}$. The corresponding theoretical analysis of the optimal error estimates is shown in the following sections, employing a technique which is different from those used previously. Moreover, as shown in Theorem 4.4 and discussed in Remark 4.2 below, the polynomially graded mesh Π_n^p is efficient for integral equations on the real line.

(ii) The iterative graded mesh Π_n^i and the exponentially graded mesh Π_n^e are both new.

(iii) Compared with (3.2), the exponentially graded mesh Π_n^e has much less mesh points further away from 0. This will save much computation time in use for unbounded domain computation problems. Moreover, the optimal order of approximation is true for solutions decaying only polynomially at infinity, as seen from Lemma 3.3 below. The results obtained in this paper do not require the solution to decay exponentially at infinity. The exponential decay at infinity of the solution was, however, required for the mesh (3.2) in the previous papers.

Lemma 3.1. *Let $h_i^{(n)} = S_{i+1}^{(n)} - S_i^{(n)}$. Then*

$$h_i^{(n)} \leqslant \frac{1}{\alpha n}\left(1 + S_{i+1}^{(n)}\right), \quad i = 0, 1, \ldots \tag{3.3}$$

for the case of exponentially graded mesh Π_n^e and

$$h_i^{(n)} \leqslant \frac{1}{\alpha n}\xi_{i+1}^{q-1}, \quad i = 0, 1, \ldots \tag{3.4}$$

for the case of polynomially graded mesh Π_n^p, where $q \geqslant 1$ and $\xi_{i+1} = 1 + (i+1)/(q\alpha n)$.

Proof. For the exponentially graded mesh Π_n^e it follows by applying the Taylor theorem to the function $g(s) = e^s - 1$ that

$$h_i^{(n)} = S_{i+1}^{(n)} - S_i^{(n)} = \left(\frac{i+1}{\alpha n} - \frac{i}{\alpha n}\right)g'(\xi) = \frac{1}{\alpha n}g'(\xi),$$

where

$$\frac{i}{\alpha n} \leqslant \xi \leqslant \frac{i+1}{\alpha n}.$$

The inequality (3.3) then follows easily on noting the fact that $g'(s) = e^s$ is an increasing function.

Similarly, applying the Taylor theorem to $(1+s)^q$, it follows that for the polynomially graded mesh Π_n^p,

$$h_i^{(n)} = S_{i+1}^{(n)} - S_i^{(n)} = \xi_{i+1}^q - \xi_i^q \leqslant \frac{1}{\alpha n}\xi_{i+1}^{q-1},$$

where use has been made of the fact that $(1+s)^{q-1}$ is an increasing function of $s > 0$ if $q \geqslant 1$. The proof is thus complete. □

We now study the approximation properties of the interpolation projection operator P_n defined in Section 2 with the above meshes. We first have the following general result on the interpolation approximation error estimate of P_n which follows from the general interpolation approximation theory.

Lemma 3.2. *Let P_n be the interpolation projective operator defined in* (2.1)*. If $\phi \in W^{r,\infty}(\mathbb{R})$, then*

$$\left\|(P_n - I)\phi\right\|_{L^\infty(I_{\pm i}^{(n)})} \leqslant M_1\left(h_{\pm i}^{(n)}\right)^l \left\|\phi^{(l)}\right\|_{L^\infty(I_{\pm i}^{(n)})}, \quad 0 \leqslant l \leqslant r,\ i = 0, 1, \ldots. \tag{3.5}$$

Applying Lemma 3.2 to the graded meshes Π_n^i, Π_n^e, and Π_n^p, we have the following interpolation approximation error estimates for the above graded meshes.

Lemma 3.3.

(i) *If, for $0 \leqslant l \leqslant r$, $\phi^{(l)} \in BC_p(\mathbb{R})$ for some $p > 0$, then for the iterative graded mesh Π_n^i with $0 < q_{\pm i} \leqslant p/r$ we have*

$$\left\|(P_n - I)\phi\right\|_{L^\infty(I_{\pm i}^{(n)})} \leqslant M_2 n^{-r} \left\|\phi^{(r)}\right\|_{BC_p(I_{\pm i}^{(n)})}, \quad i = 0, 1, \ldots, \tag{3.6}$$

where M_2 is a positive constant independent of n, i, and $\phi(s)$. Further, if, for $0 \leqslant l \leqslant r$, $\phi^{(l)} \in BC_{p+l}(\mathbb{R})$ for some $p > 0$, then for the iterative graded mesh Π_n^i with $0 < q_{\pm i} \leqslant p/r + 1$ we have

$$\left\|(P_n - I)\phi\right\|_{L^\infty(I_{\pm i}^{(n)})} \leqslant M_3 n^{-r} \left\|\phi^{(r)}\right\|_{BC_{p+r}(I_{\pm i}^{(n)})}, \quad i = 0, 1, \ldots, \tag{3.7}$$

where M_3 is independent of n, i, and $\phi(s)$.

(ii) *If, for $0 \leqslant l \leqslant r$, $\phi^{(l)} \in BC_p(\mathbb{R})$ for some $p \geqslant r$, then for the exponentially graded mesh Π_n^e we have*

$$\left\|(P_n - I)\phi\right\|_{L^\infty(I_{\pm i}^{(n)})} \leqslant M_4(\alpha) n^{-r} \left\|\phi^{(r)}\right\|_{BC_p(I_{\pm i}^{(n)})}, \quad i = 0, 1, \ldots, \tag{3.8}$$

where $M_4(\alpha)$ is independent of n, i, and $\phi(s)$. Further, if, for $0 \leqslant l \leqslant r$, $\phi^{(l)} \in BC_{p+l}(\mathbb{R})$, then for the exponentially graded mesh Π_n^e the estimate (3.7) *holds for all $p > 0$.*

(iii) *If, for $0 \leqslant l \leqslant r$, $\phi^{(l)} \in BC_p(\mathbb{R})$ for $0 < p < r$, then for the polynomially graded mesh Π_n^p with $1 \leqslant q \leqslant r/(r-p)$ the estimate* (3.8) *holds. Further, if, for $0 \leqslant l \leqslant r$, $\phi^{(l)} \in BC_{p+l}(\mathbb{R})$ for $p > 0$, then for the polynomially graded mesh Π_n^p the estimate* (3.7) *holds for all $q \geqslant 1$.*

Proof. (i) From the definition of the iterative graded mesh Π_n^i it follows that

$$h_{\pm i}^{(n)} = h\left(1 + \left|S_{\pm i}^{(n)}\right|\right)^{q_{\pm i}}, \qquad h = \frac{1}{n}.$$

So by Lemma 3.2 it is easy to see that

$$\begin{aligned}\left\|(P_n - I)\phi\right\|_{L^\infty(I_{\pm i}^{(n)})} &\leqslant M_1\left(h_{\pm i}^{(n)}\right)^r \left(1 + \left|S_{\pm i}^{(n)}\right|\right)^{-p} \left\|\phi^{(r)}\right\|_{BC_p(I_{\pm i}^{(n)})} \\ &\leqslant M_1 h^r \left(1 + \left|S_{\pm i}^{(n)}\right|\right)^{r q_{\pm i} - p} \left\|\phi^{(r)}\right\|_{BC_p(I_{\pm i}^{(n)})}.\end{aligned}$$

The estimate (3.6) then follows by noting that $0 < q_{\pm i} \leqslant p/r$. The estimate (3.7) can be derived similarly.

(ii) From Lemma 3.2 together with (3.3) it follows that

$$\begin{aligned}\big\|(P_n - I)\phi\big\|_{L^\infty(I_{\pm i}^{(n)})} &\leqslant M_1\big(h_{\pm i}^{(n)}\big)^r\big(1+S_i^{(n)}\big)^{-p}\big\|\phi^{(r)}\big\|_{C_p(I_{\pm i}^{(n)})}\\ &\leqslant M_1\frac{1}{\alpha^r n^r}\big(1+S_{i+1}^{(n)}\big)^r\big(1+S_i^{(n)}\big)^{-p}\big\|\phi^{(r)}\big\|_{BC_p(I_{\pm i}^{(n)})}.\end{aligned}$$

Since, by the definition of the exponentially graded mesh Π_n^e and noting that $p \geqslant r$, we have

$$\begin{aligned}\big(1+S_{i+1}^{(n)}\big)^r\big(1+S_i^{(n)}\big)^{-p} &\leqslant \left(\frac{1+S_{i+1}^{(n)}}{1+S_i^{(n)}}\right)^r\big(1+S_i^{(n)}\big)^{r-p}\\ &\leqslant e^{((i+1)/(\alpha n)-i/(\alpha n))r} \leqslant e^{r/\alpha}\end{aligned}$$

for all $n \geqslant 1$, then the estimate (3.8) follows with $M_3 = M_1 e^{r/\alpha}/\alpha^r$.

The second conclusion can be shown similarly.

(iii) From Lemma 3.2 in conjunction with (3.4) it is easy to see that for $q \geqslant 1$,

$$\begin{aligned}\big\|(P_n - I)\phi\big\|_{L^\infty(I_i^{(n)})} &\leqslant M_1\big(h_i^{(n)}\big)^r\big(1+S_i^{(n)}\big)^{-p}\big\|\phi^r\big\|_{BC_p(I_i^{(n)})}\\ &\leqslant M_1\frac{1}{\alpha^r n^r}\xi_{i+1}^{(q-1)r}\big(1+S_i^{(n)}\big)^{-p}\big\|\phi^{(r)}\big\|_{BC_p(I_i^{(n)})},\end{aligned} \tag{3.9}$$

where $\xi_i = 1 + i/(q\alpha n)$. By the definition of $S_i^{(n)}$ it follows that

$$\xi_{i+1}^{(q-1)r}\big(1+S_i^{(n)}\big)^{-p} = \left(\frac{\xi_{i+1}}{\xi_i}\right)^{(q-1)r}\xi_i^{(q-1)r-qp}. \tag{3.10}$$

Since $1 \leqslant q \leqslant r/(r-p)$, so $(q-1)r \geqslant 0$ and $(q-1)r - pq \leqslant 0$, then $\xi_i^{(q-1)r-pq} \leqslant 1$ for $i = 0, 1, \ldots$ and

$$\begin{aligned}\left(\frac{\xi_{i+1}}{\xi_i}\right)^{(q-1)r} &= \left(\frac{1+\frac{i+1}{q\alpha n}}{1+\frac{i}{q\alpha n}}\right)^{(q-1)r} = \left(1+\frac{\frac{1}{q\alpha n}}{1+\frac{i}{q\alpha n}}\right)^{(q-1)r}\\ &\leqslant \left(1+\frac{1}{q\alpha n}\right)^{(q-1)r} \leqslant \left(1+\frac{1}{q\alpha}\right)^{(q-1)r} := C_\alpha\end{aligned}$$

for $n \geqslant 1$. Thus the required estimate (3.8) on $I_i^{(n)}$ follows from (3.9) with $M_4(\alpha) = M_1 C_\alpha/\alpha^r$. The result on $I_{-i}^{(n)}$ can be derived similarly.

Arguing similarly as above we can prove the second conclusion. □

Remark 3.3. The constant $M_4(\alpha)$ depends on $\alpha > 0$ and may become very small if α is chosen to be very large. For simplicity, α can be taken to be 1 in practical computation.

4. Error estimate for the case $\phi(s) \in BC_p^r(\mathbb{R})$

In this section we will derive the error estimates in $L^\infty(\mathbb{R})$-norm for the numerical scheme (2.2) based on the new graded meshes introduced in the last section, under the condition that $\phi \in BC_p^r(\mathbb{R})$, where ϕ is the solution to the integral equation (1.1) or equivalently (1.2).

We first consider the standard mesh Π_n^u.

Lemma 4.1. *For the standard mesh Π_n^u we have*

$$\sup_{n>N,\, A>A_0,\, Z\in L^Q} \left\|(I - K_{z,A}P_n)^{-1}\right\| = C_0 < \infty \tag{4.1}$$

for sufficiently large $N > 0$ and $A_0 > 0$.

Proof. By the definition of Π_n^u and P_n, it is easy to verify that P_n satisfies Assumption A_2 in [12, p. 526]. So, by [12, Theorem 4.9], (4.1) holds. This completes the proof. □

Theorem 4.1. *Let ϕ_n be the approximation solution of the numerical scheme* (2.2) *based on the standard mesh Π_n^u. Choose $A_n = O(n^{r/p})$ in* (2.2). *Then there is a sufficiently large $N_1 > 0$ such that for $n > N_1$,*

$$\|\phi - \phi_n\|_{L^\infty(\mathbb{R})} = O\big(n^{-r}\big). \tag{4.2}$$

Proof. From (1.2) and (1.5) it is easy to derive that

$$(I - K_{z,A_n})(\phi - \phi_{A_n}) = (K_z - K_{z,A_n})\phi.$$

By Theorem 1.2, we have

$$\sup_{z\in L^Q,\, A\geqslant A_0} \left\|(I - K_{z,A})^{-1}\right\| = C_1 < \infty$$

for large $A_0 > 0$. Thus, and since $A_n = O(n^{r/p})$, there is an $N_1 > 0$ such that for $n > N_1$,

$$\|\phi - \phi_{A_n}\|_{L^\infty(\mathbb{R})} \leqslant C_1 \left\|(K_z - K_{z,A_n})\phi\right\|_{L^\infty(\mathbb{R})}. \tag{4.3}$$

Now by the definition of operator K_z and K_{z,A_n} it follows that

$$\begin{aligned} &\left\|(K_z - K_{z,A_n})\phi\right\|_{L^\infty(\mathbb{R})} \\ &\quad\leqslant \sup_{s\in\mathbb{R}} \left[\int_{A_n}^{+\infty} \left|\kappa(s-t)z(t)\phi(t)\right| dt + \int_{-\infty}^{-A_n} \left|\kappa(s-t)z(t)\phi(t)\right| dt \right] \\ &\quad\leqslant \|K_z\| \sup_{|t|\geqslant A_n} \left|\phi(t)\right| = O\big((1+A_n)^{-p}\big) = O\big(n^{-r}\big), \end{aligned}$$

so

$$\|\phi - \phi_{A_n}\|_{L^\infty(\mathbb{R})} = O\big(n^{-r}\big). \tag{4.4}$$

We now estimate $\|\phi_{A_n} - \phi_n\|_{L^\infty(\mathbb{R})}$. From (1.5) and (2.3) it follows that

$$\phi_{A_n} - \phi_n = (I - K_{z,A_n} P_n)^{-1}\big(K_{z,A_n}(I - P_n)\phi_{A_n}\big), \tag{4.5}$$

which together with Lemma 4.1 implies that

$$\begin{aligned}\|\phi_{A_n} - \phi_n\|_{L^\infty(\mathbb{R})} &\leqslant C_0\big\|K_{z,A_n}(I - P_n)\phi_{A_n}\big\|_{L^\infty(\mathbb{R})} \\ &\leqslant C_0\big(\big\|K_{z,A_n}(I - P_n)\phi\big\|_{L^\infty(\mathbb{R})} \\ &\quad + \big\|K_{z,A_n}(I - P_n)(\phi - \phi_{A_n})\big\|_{L^\infty(\mathbb{R})}\big).\end{aligned} \tag{4.6}$$

From (4.6) and making use of (4.4) and Lemma 3.2, we obtain, on noting that $h_{\pm i}^{(n)} = h = 1/n$, that

$$\|\phi_{A_n} - \phi_n\|_{L^\infty(\mathbb{R})} = O\big(n^{-r}\big). \tag{4.7}$$

Combining (4.4) and (4.7) leads to the required result (4.2). The theorem is thus proved. □

We now consider the iterative graded mesh Π_n^i. Take $A_n = n^{r/p}$ in (2.2). Denote by $m(n) - 1$ the index of the largest node $S_{m(n)-1}^{(n)}$ in $(-A_n, A_n)$ satisfying that $S_{m(n)}^{(n)} \geqslant A_n > S_{m(n)-1}^{(n)}$ and let $S_{\pm m(n)}^{(n)} = \pm A_n$.

Lemma 4.2. *Let $A_n = n^{r/p}$ and $S_{\pm m(n)}^{(n)} = \pm A_n$. Let*

$$0 < q_{\pm i} \leqslant \max\left((1-\varepsilon^{**})\frac{p}{r}, (1-\varepsilon^*)\frac{\ln n}{\ln(1 + |S_{\pm i}^{(n)}|)}\right)$$

for $i = 0, 1, \ldots, m(n) - 1$, where $m(n)$ is as defined above and ε^* and ε^{**} are two small positive constants. Then for the iterative graded mesh Π_n^i there is an integer $N_2 > 0$ such that for $n > N_2$,

$$\sup_{n \geqslant N_2,\, z \in L^Q} \big\|(I - K_{z,A_n} P_n)^{-1}\big\| \leqslant C_2 < \infty. \tag{4.8}$$

Proof. Note first that

$$\begin{aligned}(I - K_{z,A_n} P_n)^{-1} = {} & \big[I - K_{z,A_n} + (K_{z,A_n} - K_{z,A_n} P_n) K_{z,A_n} P_n\big]^{-1} \\ & \cdot (I - K_{z,A_n} + K_{z,A_n} P_n).\end{aligned}$$

Since K_{z,A_n} and P_n are uniformly bounded, we only need to show that

$$\sup_{n \geqslant N_2} \big\|\big(I - K_{z,A_n} + (K_{z,A_n} - K_{z,A_n} P_n) K_{z,A_n} P_n\big)^{-1}\big\| \leqslant M$$

for sufficiently large $N_2 > 0$ and all $z \in L^Q$. To do so, by (4.3) we only need to prove that

$$\sup_{n > N_2} \big\|(K_{z,A_n} - K_{z,A_n} P_n) K_{z,A_n} P_n\big\| \leqslant \frac{\delta_0}{C_1} \tag{4.9}$$

for some small $0 < \delta_0 < 1$.

For $\phi \in BC(\mathbb{R}) + V_r(\Pi_n^i)$, let $\varphi(s) = K_{z,A_n} P_n \phi$. Then

$$\begin{aligned}\bigl|K_{z,A_n}(I-P_n)\varphi(s)\bigr| &\leqslant \left|\sum_{i=0}^{m(n)-1}\int_{S_i^{(n)}}^{S_{i+1}^{(n)}} \kappa(s-t)z(t)\left(\varphi(t)-\sum_{j=1}^{r} l_{ij}^{(n)}(t)\varphi\bigl(S_{ij}^{(n)}\bigr)\right)dt\right| \\ &\quad + \left|\sum_{i=0}^{m(n)-1}\int_{S_{-(i+1)}^{(n)}}^{S_{-i}^{(n)}} \kappa(s-t)z(t)\left(\varphi(t)-\sum_{j=1}^{r} l_{-ij}^{(n)}(t)\varphi\bigl(S_{-ij}^{(n)}\bigr)\right)dt\right| \\ &\leqslant M\sum_{i=0}^{m(n)-1}\Biggl[\omega\bigl(\varphi, I_i^{(n)}, \delta_i^{(n)}\bigr)\int_{S_i^{(n)}}^{S_{i+1}^{(n)}}\bigl|\kappa(s-t)z(t)\bigr|\,dt \\ &\quad + \omega\bigl(\varphi, I_{-i}^{(n)}, \delta_{-i}^{(n)}\bigr)\int_{S_{-(i+1)}^{(n)}}^{S_{-i}^{(n)}}\bigl|\kappa(s-t)z(t)\bigr|\,dt\Biggr],\end{aligned} \tag{4.10}$$

where $\delta_{\pm i}^{(n)} \leqslant h_{\pm i}^{(n)}$ and

$$\omega(\varphi, I, \delta) = \sup\bigl\{\bigl|\varphi(s') - \varphi(s)\bigr| \bigm| s', s \in I,\ |s'-s| < \delta\bigr\}.$$

From the definition of the iterative graded mesh it follows that if

$$0 < q_{\pm i} \leqslant (1-\varepsilon^*)\frac{\ln n}{\ln(1+|S_{\pm i}^{(n)}|)},$$

then $h_{\pm i}^{(n)} = h(1+|S_{\pm i}^{(n)}|)^{q_{\pm i}} \leqslant hh^{-1+\varepsilon^*} = h^{\varepsilon^*}$ and, if $0 < q_{\pm i} \leqslant (1-\varepsilon^{**})p/r$, then

$$\begin{aligned}h_{\pm i}^{(n)} &= h\bigl(1+\bigl|S_{\pm i}^{(n)}\bigr|\bigr)^{q_i} \leqslant h\bigl(1+\bigl|S_{\pm m(n)}^{(n)}\bigr|\bigr)^{q_i} \leqslant h\bigl(1+n^{r/p}\bigr)^{q_i} \\ &\leqslant h^{1-(rq_i)/p}\bigl(1+n^{-r/p}\bigr)^{q_i} \leqslant 2h^{1-(rq_i)/p} \leqslant 2h^{\varepsilon^{**}}\end{aligned}$$

for sufficiently large n. Thus letting $\delta^{(n)} = \max(h^{\varepsilon^*}, 2h^{\varepsilon^{**}})$, we have

$$\omega\bigl(\varphi, I_{\pm i}^{(n)}, \delta_{\pm i}^{(n)}\bigr) \leqslant \omega\bigl(\varphi, I_{\pm i}^{(n)}, \delta^{(n)}\bigr)$$

and

$$\begin{aligned}\bigl|K_{z,A_n}(I-P_n)\varphi(s)\bigr| &\leqslant M \max_{0\leqslant i\leqslant m(n)-1}\omega\bigl(\varphi, I_{\pm i}^{(n)}, \delta^{(n)}\bigr)\int_{-A_n}^{A_n}\bigl|\kappa(s-t)z(t)\bigr|\,dt \\ &\leqslant M\|K_z\| \max_{0\leqslant i\leqslant m(n)-1}\omega\bigl(\varphi, I_{\pm i}^{(n)}, \delta^{(n)}\bigr).\end{aligned} \tag{4.11}$$

Since $\varphi(s) = K_{z,A_n} P_n \phi$, we have

$$\left|\varphi(s') - \varphi(s)\right| \leqslant \left| \int_{-A_n}^{A_n} \left|\kappa(s'-t) - \kappa(s-t)\right| \left|z(t)\right| |P_n\phi| \, dt \right|$$
$$\leqslant \int_{-\infty}^{+\infty} \left|\kappa(s'-t) - \kappa(s-t)\right| dt \, \|z\|_{L^\infty(\mathbb{R})} \|P_n\| \|\phi\|_{L^\infty(\mathbb{R})}.$$

Thus, and since $\|z\|_{L^\infty(\mathbb{R})} \leqslant M$ for all $z \in L^Q$ and $\|P_n\| \leqslant M$ for all $n \geqslant 1$, we obtain from (4.11) that

$$\left\|K_{z,A_n}(I - P_n)K_{z,A_n}P_n\right\| \leqslant M \sup_{|s'-s|<\delta^{(n)}} \int_{-\infty}^{+\infty} \left|\kappa(s'-t) - \kappa(s-t)\right| dt. \tag{4.12}$$

Since $\kappa \in L^1(\mathbb{R})$, it holds that the right-hand side of the above inequality goes to zero if $\delta^{(n)} \to 0$. Now $\varepsilon^*, \varepsilon^{**} > 0$ and $h = 1/n$ so $\delta^{(n)} \to 0$ as $n \to \infty$. The inequality (4.9) thus follows from (4.12). The lemma is thus proved. □

Theorem 4.2. *Let ϕ_n be the approximation solution of the numerical scheme* (2.2) *based on the iterative graded mesh Π_n^i with $A_n = n^{r/p}$ and $S_{\pm m(n)}^{(n)} = \pm A_n$ where $m(n)$ is as defined above. Choose $q_{\pm i}$ so that $0 < q_{\pm i} \leqslant (1 - \varepsilon^{**})p/r$ with $\varepsilon^{**} > 0$ being a small constant. Then, for sufficiently large n,*

$$\|\phi - \phi_n\|_{L^\infty(\mathbb{R})} = O\left(n^{-r}\right). \tag{4.13}$$

Proof. Similarly to the proof of Theorem 4.1, it can be deduced that

$$\left\|(K_z - K_{z,A_n})\phi\right\|_{L^\infty(\mathbb{R})}$$
$$\leqslant \sup_{s\in\mathbb{R}} \left[\int_{A_n}^{+\infty} \left|\kappa(s-t)z(t)\phi(t)\right| dt + \int_{-\infty}^{-A_n} \left|\kappa(s-t)z(t)\phi(t)\right| dt \right]$$
$$\leqslant \|K_z\| \sup_{|t|\geqslant A_n} \left|\phi(t)\right| = O\left((1 + A_n)^{-p}\right) = O\left(n^{-r}\right),$$

so, by using (4.3), we have

$$\|\phi - \phi_{A_n}\|_{L^\infty(\mathbb{R})} = O\left(n^{-r}\right). \tag{4.14}$$

On the other hand, by (4.5) and Lemma 4.2, it is easy to see that

$$\|\phi_{A_n} - \phi_n\|_{L^\infty(\mathbb{R})} \leqslant C_2 \left\|K_{z,A_n}(I - P_n)\phi_{A_n}\right\|_{L^\infty(\mathbb{R})}$$
$$\leqslant C_2\left[\left\|K_{z,A_n}(I - P_n)\phi\right\|_{L^\infty(\mathbb{R})} + \left\|K_{z,A_n}(I - P_n)(\phi_{A_n} - \phi)\right\|_{L^\infty(\mathbb{R})}\right]. \tag{4.15}$$

Applying Lemma 3.3(i) and noting that $0 < q_{\pm i} < p/r$, we obtain that

$$\begin{aligned}\|K_{z,A_n}(I-P_n)\phi\|_{L^\infty(\mathbb{R})} &= \left\| \int_{-A_n}^{A_n} \left|\kappa(s-t)z(t)(I-P_n)\phi(t)\right| dt \right\|_{L^\infty(\mathbb{R})} \\ &\leqslant \|K_z\| \left\|(I-P_n)\phi\right\|_{L^\infty([-A_n,A_n])} \\ &\leqslant \|K_z\| \max_{0\leqslant i\leqslant m(n)-1} \left\|(I-P_n)\phi\right\|_{L^\infty(I_{\pm i}^{(n)})} = O\left(n^{-r}\right).\end{aligned}$$

This together with (4.14) and (4.15) implies that

$$\|x_{A_n} - x_n\|_{L^\infty(\mathbb{R})} \leqslant O\left(n^{-r}\right). \tag{4.16}$$

Combining (4.14) and (4.16) gives the required result (4.13). The proof is thus complete. □

For $p \geqslant r$, we use the exponentially graded mesh Π_n^e. Take $A_n = n^{r/p}$ in (2.2), denote by $m(n) - 1$ the index of the largest node in $(-A_n, A_n)$ satisfying that $S_{m(n)}^{(n)} \geqslant n^{r/p} > S_{m(n)-1}^{(n)}$ and let $S_{\pm m(n)}^{(n)} = \pm A_n$. Note that $e^{m(n)/(\alpha n)} - 1 \approx n^{r/p}$ so $m(n) \approx \alpha n \ln(n^{r/p} + 1)$.

Lemma 4.3. *If $p \geqslant r$, $A_n = n^{r/p}$, and $S_{\pm m(n)}^{(n)} = \pm A_n$ with $m(n)$ being defined as above, then for the exponentially graded mesh Π_n^e with $\alpha > 0$ in the case $p > r$ or $\alpha \geqslant \alpha_0 > 0$ in the case $p = r$ for some large α_0 it holds that*

$$\sup_{n\geqslant N_3,\, z\in L^Q} \left\|(I - K_{z,A_n}P_n)^{-1}\right\| \leqslant C_3 < \infty \tag{4.17}$$

for some sufficiently large $N_3 > 0$.

Proof. Similarly as in the proof of Lemma 4.1, it is enough to prove that for some small δ_0 with $0 < \delta_0 < 1$ there is either an $N_3 > 0$ in the case $p > r$ or an $\alpha_0 > 0$ in the case $p = r$ such that

$$\left\|K_{z,A_n}(I - P_n)K_{z,A_n}P_n\right\| \leqslant \frac{\delta_0}{C_1} \tag{4.18}$$

for all $n > N_3$ and all $\alpha > 0$ in the case $p > r$ or for all $n \geqslant 1$ and all $\alpha \geqslant \alpha_0$ in the case $p = r$.

First, noting the definition of A_n, $m(n)$, and applying Lemma 3.1, we have that for $0 \leqslant i \leqslant m(n) - 1$,

$$h_i^{(n)} \leqslant \frac{1}{\alpha n}\left(1 + S_{m(n)}^{(n)}\right) = \frac{1}{\alpha n}\left(1 + n^{r/p}\right).$$

Thus, if $p > r$, then

$$h_i^{(n)} \leqslant \frac{1}{\alpha n} n^{r/p}\left(1 + n^{-r/p}\right) = \frac{2}{\alpha} n^{-(1-r/p)},$$

and if $p = r$, then

$$h_i^{(n)} \leqslant \frac{1+n}{\alpha n} \leqslant \frac{2}{\alpha}.$$

Now for $p > r$ let $\delta^{(n)} = (2/\alpha)n^{-(1-r/p)}$ and for $p = r$ let $\delta^{(n)} = 2/\alpha$. Then arguing in the same way as in the proof of Lemma 4.1, it can be obtained that

$$\left\| K_{z,A_n}(I-P_n)K_{z,A_n}P_n \right\| \leqslant M \sup_{|s'-s|<\delta^{(n)}} \int_{-\infty}^{+\infty} \left|\kappa(s'-t)-\kappa(s-t)\right| dt.$$

From this and the definition of $\delta^{(n)}$ it follows that (4.18) is true. The lemma is thus proved by noting the remark at the beginning of the proof. □

Similar argument as in the proof of Theorem 4.2 using Lemmas 4.3 and 3.3(ii) leads to the following theorem.

Theorem 4.3. *Let $p \geqslant r$. Let ϕ_n be the approximation solution of the numerical scheme* (2.2) *based on the exponentially graded mesh Π_n^e with $\alpha \geqslant \alpha_0$ for some large $\alpha_0 >$ in the case $p = r$. Let $A_n = n^{r/p}$ and $S_{\pm m(n)}^{(n)} = \pm A_n$, where $m(n)$ is the same as in Lemma* 4.3. *Then, for sufficiently large n,*

$$\|\phi - \phi_n\|_{L^\infty(\mathbb{R})} = O\left(n^{-r}\right). \tag{4.19}$$

Now, for the case $p < r$, we consider the polynomially graded mesh Π_n^p. Take $A_n = n^{r/p}$ in (2.2), let $m(n)$ be the index of the node such that $S_{m(n)}^{(n)} \geqslant A_n > S_{m(n)-1}^{(n)}$ and choose $S_{\pm m(n)}^{(n)} = \pm A_n$. Note that $(1 + m(n)/(q\alpha n))^q - 1 \approx n^{r/p}$ so

$$m(n) \approx q\alpha n\left(\left(n^{r/p}+1\right)^{1/q} - 1\right) \approx q\alpha n^{1+r/(pq)} \quad \text{as } n \to \infty.$$

In particular, if $q = r/(r-p)$ then $m(n) \approx q\alpha n^{r/p}$.

Lemma 4.4. *Let $p < r$ and let $A_n = n^{r/p}$ and $S_{\pm m(n)}^{(n)} = \pm A_n$. Then for the polynomially graded mesh Π_n^p with $1 \leqslant q \leqslant r/(r-p)$ it holds in the case $1 \leqslant q < r/(r-p)$ that for all $\alpha > 0$*

$$\sup_{n \geqslant N_4,\, z \in L^Q} \left\|(I - K_{z,A_n}P_n)^{-1}\right\| \leqslant C_4 < \infty \tag{4.20}$$

for some sufficiently large $N_4 > 0$ or it holds in the case $q = r/(r-p)$ that (4.20) *is true for all $\alpha \geqslant \alpha_0$ with $\alpha_0 > 0$ large enough.*

Proof. Similarly as in the proof of Lemma 4.1, it is enough to prove that for some small δ_0 with $0 < \delta_0 < 1$ there is either an $N_4 > 0$ in the case $1 \leqslant q < r/(r-p)$ or an $\alpha_0 > 0$ in the case $q = r/(r-p)$ such that

$$\left\| K_{z,A_n}(I-P_n)K_{z,A_n}P_n \right\| \leqslant \frac{\delta_0}{C_1} \tag{4.21}$$

for all $n > N_4$ and all $\alpha > 0$ in the case $1 \leqslant q < r/(r-p)$ or for all $n \geqslant 1$ and all $\alpha \geqslant \alpha_0$ in the case $q = r/(r-p)$.

Note first that from Lemma 3.1 and the definition of A_n and $m(n)$ it follows that for $0 \leqslant i \leqslant m(n) - 1$,

$$h_i^{(n)} \leqslant \frac{1}{\alpha n}\xi_{i+1}^{q-1} \leqslant \frac{1}{\alpha n}\left(\xi_{i+1}^{q}\right)^{(q-1)/q},$$

where $\xi_i = 1 + i/(\alpha q n)$. By the definition of $S_i^{(n)}$ in the polynomially graded mesh and noting that $1 \leqslant q \leqslant r/(r-p)$, we have that for $0 \leqslant i \leqslant m(n) - 1$,

$$h_i^{(n)} \leqslant \frac{1}{\alpha n}\left(1 + S_{i+1}^{(n)}\right)^{\frac{q-1}{q}} \leqslant \frac{1}{\alpha n}\left(n^{r/p} + 1\right)^{\frac{q-1}{q}} \leqslant \frac{1}{\alpha n} n^{\frac{r(q-1)}{pq}}\left(1 + n^{-r/p}\right)^{\frac{(q-1)}{q}}$$
$$\leqslant \frac{2}{\alpha} n^{-(1-\frac{r(q-1)}{pq})}.$$

Now let

$$\delta^{(n)} = \frac{2}{\alpha} n^{-(1-\frac{r(q-1)}{pq})}.$$

Then arguing in exactly the same way as in the proof of Lemma 4.1, it can be derived that

$$\left\| K_{z,A_n}(I - P_n)K_{z,A_n}P_n \right\| \leqslant M \sup_{|s'-s|<\delta^{(n)}} \int_{-\infty}^{+\infty} \left|\kappa(s'-t) - \kappa(s-t)\right| dt.$$

From this and the definition of $\delta^{(n)}$ it follows that (4.21) is true in both cases. Noting the remark at the beginning of the proof completes the proof. □

From Lemmas 4.4 and 3.3(iii) the following theorem can be easily obtained by arguing similarly as in the proof of Theorem 4.2.

Theorem 4.4. *Let* $0 < p < r$. *Let* ϕ_n *be the approximation solution of the numerical scheme* (2.2) *by using the polynomially graded mesh* Π_n^p *with* $1 \leqslant q \leqslant r/(r-p)$. *Let* $\alpha \geqslant \alpha_0$ *for some large* $\alpha_0 >$ *in the case* $q = r/(r-p)$. *Take* $A_n = n^{r/p}$ *and* $S_{\pm m(n)}^{(n)} = \pm A_n$ *where* $m(n)$ *is the same as in Lemma* 4.4. *Then, for sufficiently large* n,

$$\|\phi - \phi_n\|_{L^\infty(\mathbb{R})} = O\left(n^{-r}\right). \tag{4.22}$$

Remark 4.1. For the case $0 < p < r$ the best choice for q is $r/(r-p)$ in the polynomially graded mesh Π_n^p.

Remark 4.2. For the same level of accuracy the exponentially graded mesh Π_n^e requires the least mesh points ($\approx (\alpha r/p)n\ln n$) to solve the numerical scheme (2.2) and the polynomially graded mesh Π_n^p needs the second least mesh points ($\approx q\alpha n^{1+r/(pq)}$), whilst the uniform mesh Π_n^u needs the most mesh points ($\approx 2n^{r/p+1}$). Thus, if the solution ϕ of the integral equation (1.1) decays faster at infinity (e.g., $p \geqslant r$), then the exponentially graded mesh Π_n^e is the most efficient one among the four meshes. However, if the solution ϕ decays slower at infinity (e.g., $p < r$), then the polynomially graded mesh Π_n^p will be a better choice than the exponentially graded one though the exponentially graded mesh requires much less mesh points to solve (2.2). This is because the exponentially graded mesh has much less mesh points placed further away from 0 so solutions with slower decay at infinity may not be better approximated compared with the polynomially graded mesh. Similarly, if the solution ϕ decays very slow at infinity, then both the uniform mesh and the iteratively graded mesh may be more effective compared with the exponentially and polynomially graded meshes.

5. The case $\phi \in BC^r_{p+r}(\mathbb{R})$

In this section we assume that the solution ϕ to the integral equation (1.1) or equivalently (1.2) satisfies the condition $\phi \in BC^r_{p+r}(\mathbb{R})$ and establish, in this case, the error estimates in $L^\infty(\mathbb{R})$-norm for the numerical scheme (2.2) based on the new graded meshes introduced in Section 3.

Remark 5.1. If $\phi \in BC^r_{p+r}(\mathbb{R})$, then Theorems 4.1–4.4 in Section 4 remain true. However, we can further establish the following new results.

Theorem 5.1. *Let $\phi \in BC^r_{p+r}(\mathbb{R})$ with $p > 0$ be the solution to the integral equation* (1.1) *or equivalently* (1.2) *and let ϕ_n be the approximation solution of the numerical scheme* (2.2) *by using the iterative graded mesh Π^i_n with $A_n = n^{r/p}$ and $S^{(n)}_{\pm m(n)} = \pm A_n$ where $m(n) - 1$ is the index of the largest node in $(-A_n, A_n)$ satisfying that $S^{(n)}_{m(n)} \geqslant A_n > S^{(n)}_{m(n)-1}$. Choose $q_{\pm i}$ so that for $i = 0, 1, \ldots, m(n) - 1$,*

$$0 < q_{\pm i} \leqslant \min\left[\frac{p+r}{r}, \max\left((1-\varepsilon^{**})p/r, \frac{(1-\varepsilon^*)\ln n}{\ln\left(1 + \left|S^{(n)}_{\pm i}\right|\right)}\right)\right],$$

where $\varepsilon^*, \varepsilon^{**} > 0$ are two small constants. Then, for sufficiently large n,

$$\|\phi - \phi_n\|_{L^\infty(\mathbb{R})} = O\left(n^{-r}\right). \tag{5.1}$$

Proof. From Lemmas 4.2 and 3.3(i) (cf. (3.7)) the result (5.1) can be shown in exactly the same way as in the proof of Theorem 4.2. □

For $0 < p < r$ we may also consider the composite graded mesh as defined in the following theorem for the numerical scheme (2.2).

Theorem 5.2. *Let $0 < p < r$. Let $\phi \in BC^r_{p+r}(\mathbb{R})$ be the solution to the integral equation* (1.1) *or equivalently* (1.2) *and let ϕ_n be the approximation solution of the numerical scheme* (2.2) *based on the composite graded mesh: $A_n = n^{r/p}$, $S^{(n)}_{\pm m(n)} = \pm A_n$, where $m(n)$ is the index of the node such that $S^{(n)}_{m(n)} \geqslant A_n > S^{(n)}_{m(n)-1}$, and using the exponentially graded mesh Π^e_n in $[-n, n]$ and using the polynomially graded mesh Π^p_n with $1 \leqslant q \leqslant r/(r-p)$ in the intervals $[-A_n, -n]$ and $[n, A_n]$. Then, for sufficiently large n,*

$$\|\phi - \phi_n\|_{L^\infty(\mathbb{R})} = O\left(n^{-r}\right). \tag{5.2}$$

Proof. From the proof of Theorems 4.1–4.4 it is clear that we only need to prove the uniform boundedness of $\|(I - K_{z,A_n} P_n)^{-1}\|$ for the composite graded mesh. Further, from the proof of Lemmas 4.3–4.4 it is enough to show that for sufficiently large n,

$$\big\|K_{z,A_n}(I - P_n)K_{z,A_n}P_n\big\| \leqslant \frac{\delta_0}{C_1} \tag{5.3}$$

for some small $0 < \delta_0 < 1$.

In fact, arguing similarly as in deriving (4.11) (cf. the proof of Lemmas 4.3–4.4), we have

$$\begin{aligned}\big|K_{z,A_n}(I - P_n)\varphi(s)\big| \leqslant M\Big[&\max_{I^{(n)}_{\pm i}\in[-n,n]} \omega\big(\varphi, I^{(n)}_{\pm i}, \delta^{(n)}_*\big)\\ &+ \max_{I^{(n)}_{\pm i}\in[-A_n,-n]\cup[n,A_n]} \omega\big(\varphi, I^{(n)}_{\pm i}, \delta^{(n)}_{**}\big)\Big],\end{aligned}$$

where

$$\delta^{(n)}_* = \frac{2}{\alpha_e}, \qquad \delta^{(n)}_{**} = \frac{2}{\alpha_p} n^{-(1-\frac{r(q-1)}{qp})}$$

with α_e being the α defined in the exponentially graded mesh and α_p being the α defined in the polynomially graded mesh. Noting that $\|z\|_{L^\infty(\mathbb{R})} \leqslant M$ for all $z \in L^Q$ and $\|P_n\| \leqslant M$ for all $n \geqslant 1$ and using the fact that $1 \leqslant q \leqslant r/(r - p)$ and $0 < p < r$, we can find some sufficiently large N and $\alpha_0 > 0$ such that for all $n \geqslant N$ and all $\alpha_e, \alpha_p > \alpha_0$,

$$\big\|K_{z,A_n}(I - P_n)K_{z,A_n}P_n\big\| \leqslant \sup_{|s'-s|<\max(\delta^{(n)}_*,\delta^{(n)}_{**})} \int_{-\infty}^{+\infty} \big|\kappa(s' - t) - \kappa(s - t)\big|\,dt \leqslant \frac{\delta_0}{C_1},$$

that is, (5.3) holds. The theorem is thus proved. □

6. Local error estimates

In this section we establish the local error estimate $\|x - x_n\|_{L^\infty(-A,A)}$ for some $A > 0$. In practical computation it is expected that for given $A > 0$, $\|x - x_n\|_{L^\infty(-A,A)}$ would be very small. Throughout this section we assume that

(H) $\kappa(s) = O(s^{-\beta})$ for some $\beta > 1$ as $s \to \infty$.

If (H) holds and $\phi \in BC^r_p(\mathbb{R})$ for some $p > 0$, then by (1.2) and (1.5) it follows that

$$\begin{aligned}\big|(K_z - K_{z,A_n})\phi(s)\big| &= \left|\int_{A_n}^{+\infty} \kappa(s - t)z(t)\phi(t)\,dt + \int_{-\infty}^{-A_n} \kappa(s - t)z(t)\phi(t)\,dt\right|\\ &= O\big(\big(1 + A_n - |s|\big)^{1-\beta} A_n^{-p}\big).\end{aligned}$$

Thus if $A - 1 \leqslant \theta A_n$ for some $0 < \theta < 1$, then we have that for $s \in [-A, A]$,

$$\left|(K_z - K_{z,A_n})\phi(s)\right| = O\left(A_n^{1-p-\beta}\right),$$

which together with (4.3) implies that

$$\|\phi - \phi_{A_n}\|_{L^\infty([-A,A])} = O\left(A_n^{1-\beta-p}\right).$$

Thus we conclude that if we only need to get $\|\phi - \phi_{A_n}\|_{L^\infty(-A,A)} = O(n^{-r})$, then we may take $A_n = \max(n^{r/(p+\beta-1)}, (A-1)/\theta)$, which is much smaller than the choice $A_n = n^{r/p}$ in the last section.

Theorem 6.1. *Assume that* (H) *is satisfied and let* $\phi \in BC^r_p(\mathbb{R})$. *Let* ϕ_n *be the approximation solution of the numerical scheme* (2.2) *using the iterative graded mesh* Π^i_n *with* $0 < q_{\pm i} \leqslant p/r$, $A_n = \max(n^{r/(p+\beta-1)}, (A-1)/\theta)$, *and* $S^{(n)}_{\pm m(n)} = \pm A_n$, *where* $m(n)$ *is the index of the node such that* $S^{(n)}_{m(n)} \geqslant A_n > S^{(n)}_{m(n)-1}$. *Then there is an* $N_6 > 0$ *such that for all* $n \geqslant N_6$

$$\|\phi - \phi_n\|_{L^\infty([-A,A])} = O\left(n^{-r}\right). \tag{6.1}$$

Proof. If, for $i = 0, 1, \ldots, m(n) - 1$,

$$0 < q_{\pm i} \leqslant (1 - \varepsilon^{**})(p + \beta - 1)/r \tag{6.2}$$

for some small $\varepsilon^{**} > 0$, then the theorem follows immediately from Theorem 4.2. Now, since $0 \leqslant q_{\pm i} \leqslant p/r$ and $\beta > 1$, then it is easy to see that (6.2) is satisfied, which proves the theorem. □

Applying Theorem 5.1, the following result can be easily obtained.

Theorem 6.2. *Assume that* (H) *is satisfied and that* $\phi \in BC^r_{p+r}(\mathbb{R})$. *Let* ϕ_n *be the approximation solution of the numerical scheme* (2.2) *using the iterative graded mesh* Π^i_n *with* $A_n = \max(n^{r/(p+\beta-1)}, (A-1)/\theta)$ *and* $S^{(n)}_{\pm m(n)} = \pm A_n$, *where* $m(n)$ *is the same as in Theorem* 6.1. *Choose* $q_{\pm i}$ *satisfying that for* $i = 0, 1, \ldots, m(n) - 1$,

$$0 < q_{\pm i} < \min\left[\frac{p+r}{r}, \max\left((1-\varepsilon^{**})\frac{p+\beta-1}{r}, (1-\varepsilon^*)\frac{\ln n}{\ln\left(1 + \left|S^{(n)}_{\pm i}\right|\right)}\right)\right]$$

for some small $\varepsilon^* > 0$ *and* $\varepsilon^{**} > 0$. *Then the local error estimate* (6.1) *holds.*

As an immediate consequence of Remark 5.1 and Theorems 4.3 and 4.4, we have the following result.

Theorem 6.3. *Assume that* (H) *is satisfied and that* $\phi \in BC^r_{p+r}(\mathbb{R})$. *Let* ϕ_n *be the approximation solution of the numerical scheme* (2.2) *using either the exponentially graded mesh* Π^e_n *in the case when* $p + \beta - 1 \geqslant r$ *or the polynomially graded mesh* Π^p_n *in the case when* $p + \beta - 1 < r$ *with* $1 \leqslant q \leqslant r/[r - (p + \beta - 1)]$. *Let* $A_n = \max(n^{r/(p+\beta-1)}, (A-1)/\theta)$ *and* $S^{(n)}_{\pm m(n)} = \pm A_n$, *where* $m(n)$ *is the index of the node such that* $S^{(n)}_{m(n)} \geqslant A_n > S^{(n)}_{m(n)-1}$. *Then the local error estimate* (6.1) *holds.*

Remark 6.1.

(i) If we only need to get $\|\phi - \phi_n\|_{L^\infty(-A,A)} = O(n^{-r})$, then we may use a larger graded mesh with a smaller number of grid points in practical computation.
(ii) In Theorem 6.3 for the case $p + \beta - 1 < r$ we may also use the composite graded mesh introduced as in Theorem 5.2.

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