Error Estimates for Mixed Finite Element Approximations of the Viscoelasticity Wave Equation

Liping Gao[*], Dong Liang[†], and Bo Zhang[‡]

Abstract

This paper studies mixed finite element approximations to the solution of the viscoelasticity wave equation. Two new transformations are introduced and a corresponding system of first order differential-integral equations is derived. The semi-discrete and full-discrete mixed finite element methods are then proposed for the problem based on the Raviart-Thomas-Nedelec spaces. The optimal error estimates in L^2-norm are obtained for the semi-discrete and full-discrete mixed approximations of the general viscoelasticity wave equation.

Key words. Viscoelasticity, Wave equation, Mixed finite element Method, Error estimates.

AMS(MOS) subject classifications. 65M50, 65M60

1 Introduction

Let Ω be a bounded domain in R^2 with a smooth boundary, $\partial\Omega$. For $0 < T < \infty$ a fixed real number and for $J = (0, T]$, we consider the initial-boundary value problem:

$$u_{tt} - \nabla \cdot \Big(a(x)\nabla u + b(x)\nabla u_t\Big) = f(x,t), \quad x \in \Omega, t \in J, \qquad (1.1a)$$

with the initial conditions

$$u(x,0) = u_0(x), \quad x \in \Omega, \qquad (1.1b)$$

$$u_t(x,0) = u_1(x), \quad x \in \Omega, \qquad (1.1c)$$

and the homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in J, \qquad (1.1d)$$

[*]School of Mathematical and Information Sciences, Coventry University, Coventry CV1 5FB, UK (l.gao@coventry.ac.uk)

[†]Department of Mathematics and Statistics, York University, Toronto, ON M3J 1P3 Canada (dliang@mathstat.yorku.ca)

[‡]School of Mathematical and Information Sciences, Coventry University, Coventry CV1 5FB, UK (b.zhang@coventry.ac.uk)

where $u(x,t)$ represents the displacement, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $a(x)$ is the coefficient of elasticity and $b(x)$ is the coefficient of viscoelasticity, which satisfy the property that there exist positive constants $a_i, b_i (i = 0, 1)$ such that

$$0 < a_0 \le a(x) \le a_1, \quad 0 < b_0 \le b(x) \le b_1, \quad x \in \bar{\Omega}.$$

Additionally, the coefficients $a(x)$ and $b(x)$, the initial displacement functions $u_0(x)$ and $u_1(x)$, and the force function $f(x,t)$ are assumed as regular as necessary. The equation (1.1) is called a viscoelasticity type wave equation, which describes the wave propagation phenomena of actual vibration through a viscoelastic medium.

Numerical methods for the problem (1.1) with the case $b(x) = \gamma a(x)$ (or the coefficient matrix $B(x) = \gamma A(x)$), where $\gamma > 0$ is a constant, have been developed and analyzed successfully in [7] [10] [11] [15]. Previously, Larsson and Thomée in [10] (1991) proposed a finite element method to (1.1) with the case $B(x) = \gamma A(x)$ to approximate the displacement and velocity. Both semi-discrete and fully discrete schemes were discussed, and the optimal error bounds were derived. Lin, Thomée, and Wahlbin ([11], 1991) studied the problem based on a parabolic integro-differential equation (and the viscoelasticity type equation). The new projection of Ritz-Volterra type was introduced and the optimal error estimates of the semi-discrete finite element approximation were obtained for the problems. Recently, Pani and Yuan ([15], 2001) discussed the mixed finite element method for the problem (1.1) with the case $B(x) = \gamma A(x)$. In this case, one can introduce the stress, i.e, $\sigma = A\nabla u$, then the equation can be transformed into a system of differential equations. The corresponding system is then approximated by the mixed finite element method. Optimal error estimates in L^2-norm for both the velocity and stress are derived using the usual energy argument in [15]. However, for the general case with $b(x) \neq \gamma a(x)$, the mixed finite element method can not directly be applied, which will leads to a very complex mixed system. The numerical computation and the theoretical analysis will become much complex and even impossible. Therefore, there is considerable interest in finding new transformations to construct the mixed finite element methods for the general problem and in analyzing the errors of the derived mixed methods for the problem.

In this paper, we consider the mixed finite element approximations of the problem (1.1) with the general coefficients $a(x)$ and $b(x)$ ($b(x) \neq \gamma a(x)$). In contrast with the method in [15], we introduce two new transformations: $u_t = q$ and $z = a\nabla u + b\nabla u_t$, where q and z represent the velocity and the actual stress, respectively. For constructing the mixed finite element method to the problem, a relation between q and z is derived as $\nabla q = \dfrac{1}{b} z - \dfrac{a}{b^2} e^{-\frac{a}{b}t} \displaystyle\int_0^t e^{\frac{a}{b}\tau} z \, d\tau$. Following this, a system of first order differential-integral equations is derived and the semi-discrete and full-discrete mixed finite element methods are proposed based on the Raviart-Thomas-Nedelec spaces. We prove the optimal error estimates in L^2-norm of the displacement, velocity, and actual stress for both the semi-discrete mixed scheme and the full-discrete mixed scheme.

We would also remark some previous results ([4] [5] [8] [12]) on mixed methods for the second-order wave equations without the viscoelastic term. Geveci in [8] derived the L^2-norm error bounds for the continuous-time mixed finite element approximations of velocity and stress. Cowsar, Dupont, and Wheeler in [4] [5] discussed the mixed finite element methods to a second-order hyperbolic equation with absorbing boundary conditions and other boundary conditions and derived the optimal error bounds for displacement and velocity for the semi-discrete and fully discrete schemes. Makridakis in [12] discussed the mixed methods for the linear elastodynamics problem.

The remaining part of the paper is organized as follows. In Section 2, the semi-discrete and full-discrete mixed finite element methods are formulated after introducing two new transformations. In Section 3, A non-classical projection of the solution is defined, and the approximation properties are proved. The existence and error estimates of the solution for the semi-discrete mixed scheme are established in Section 4. In Section 5, the error estimates for the full-discrete mixed scheme are discussed.

2 Mixed finite element Methods

In this section we consider the mixed finite element formulations for the problem (1.1). For the general case with $b(x) \neq \gamma a(x)$, we introduce two new transformations and transform (1.1) into a system of first order differential-integral equations. We then define the semi-discrete and full-discrete mixed approximations to the corresponding first order system of differential-integral equations.

Define

$$q = u_t, \qquad z = a\nabla u + b\nabla u_t. \tag{2.1a}$$

Solving the ordinary differential equation for ∇u, we have

$$\nabla u = \frac{1}{b(x)} e^{-\frac{a(x)}{b(x)}t} \int_0^t z(x,\tau) e^{\frac{a(x)}{b(x)}\tau} d\tau, \tag{2.1b}$$

where the right-hand side term of (2.1b) is an integral of a vector function. Differentiating (2.1b) with respect to t and using (2.1a), we obtain

$$\nabla q = \frac{1}{b} z - \frac{a}{b^2} e^{-\frac{a}{b}t} \int_0^t e^{\frac{a}{b}\tau} z d\tau. \tag{2.1c}$$

Thus, using the transformations we obtain the equivalent system of differential-integral equations for the problem (1.1):

$$q_t - div\ z = f(x,t), \quad x \in \Omega,\ t \in J, \tag{2.2a}$$

$$-\nabla q + \frac{1}{b} z = d(x,t) \int_0^t g(x,\tau) z\, d\tau, \quad x \in \Omega,\ t \in J, \tag{2.2b}$$

$$u_t - q = 0, \quad x \in \Omega,\ t \in J, \tag{2.2c}$$

with the initial values $q(x,0) = u_1(x), z(x,0) = z_0(x) := a\nabla u_0 + b\nabla u_1$, and $u(x,0) = u_0(x)$. Here we have used the notations: $d(x,t) := \dfrac{a(x)}{b(x)^2} e^{-\frac{a(x)}{b(x)}t}$ and $g(x,t) := e^{\frac{a(x)}{b(x)}t}$.

Let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm in $L^2(\Omega)$ (or $(L^2(\Omega))^2$), respectively, i.e, for any $\phi, \psi \in L^2(\Omega)$ (or $L^2(\Omega))^2$)

$$(\phi,\psi) = \int_\Omega \phi(x)\psi(x)dx, \qquad \|\phi\| = (\phi,\phi)^{\frac{1}{2}}.$$

Denote by $H^s(\Omega)$ and $(H^s(\Omega))^2$ the standard Sobolev spaces of real-valued functions and vector functions defined on Ω and by $\|\cdot\|_s$ the norms of $H^s(\Omega)$ and $(H^s(\Omega))^2$. For any space X with norm $\|\cdot\|_X$, take $L^2(0,T;X)$ to be the space of maps of $[0,T]$ into X and define the following norm for $\Phi : [0,T] \to X$:

$$\|\Phi\|^2_{L^2(0,T;X)} = \int_0^T \|\Phi(\cdot,t)\|^2_X dt, \;\; \|\Phi(\cdot,t)\|_{L^\infty(0,T;X)} = \sup_{0\le t\le T} \|\Phi(\cdot,t)\|_X.$$

Let $V = L^2(\Omega)$. Define $H(div;\Omega)$, a subspace of $(L^2(\Omega))^2$, by

$$H := H(div;\Omega) = \{\varphi \in (L^2(\Omega))^2 |\; div\varphi \in L^2(\Omega)\}$$

with the associated norm

$$\|\varphi\|^2_H = \|\varphi\|^2 + \|\, div\; \varphi\|^2.$$

Then, corresponding to (2.2), we have the mixed weak form of the problem: Find $\{q,z,u\}$: $[0,T] \to V \times H \times V$ such that

$$(q_t, v) - (div\; z, v) = (f, v), \quad \forall v \in V, \tag{2.3a}$$

$$(q, div\;\varphi) + (\frac{1}{b}z, \varphi) = (d\int_0^t gz\; d\tau, \varphi), \;\; \forall\varphi \in H, \tag{2.3b}$$

$$(u_t, v) - (q, v) = 0, \quad \forall v \in V, \tag{2.3c}$$

with the initial values $q(x,0) = u_1(x)$, $z(x,0) = z_0(x)$, and $u(x,0) = u_0(x)$.

Let $\Omega_h = \{K\}$ be the quasi-regular triangulation or rectangulation partition of the domain Ω. Denote h to be the spatial step size $h = \{\max K : K \in \Omega_h\}$. Let $V_h \times H_h \subset V \times H$ be the Raviart-Thomas-Nedelec mixed finite element spaces ([16, 14]) with index k, where k is a fixed nonnegative integer (RT_k and $RT_{[k]}$ in the notation of Brezzi and Fortin [1]). Then the semi-discrete mixed finite element approximations to (2.3) is defined as: Find $\{Q(t), Z(t), U(t)\}$: $[0,T] \to V_h \times H_h \times V_h$ such that

$$(Q_t, v_h) - (div\; Z, v_h) \;=\; (f, v_h), \quad \forall v_h \in V_h, \tag{2.4a}$$

$$(Q, div\;\varphi_h) + (\frac{1}{b}Z, \varphi_h) = (d\int_0^t gZ\; d\tau, \varphi_h), \;\; \forall\varphi_h \in H_h, \tag{2.4b}$$

$$(U_t, v_h) - (Q, v_h) = 0, \quad \forall v_h \in V_h, \tag{2.4c}$$

with initial values $U(0) \in V_h, Q(0) \in V_h, Z(0) \in H_h$, which are approximations of $u_0(x)$, $u_1(x)$, and $z_0(x)$ onto V_h, V_h, and H_h, respectively.

In order to define the full-discrete scheme, let $N>0$ be the integer number and define the time step size $\Delta t=\dfrac{T}{N}$. Then, $[0,T]$ can be divided as:

$$0=t^0<t^1<\cdots<t^n<\cdots<t^N=T,$$

with $t^n=n\Delta t$. Set

$$v^n=v(t^n),\quad v^{n+\frac{1}{2}}=\frac{1}{2}(v^n+v^{n+1}),$$

$$\partial_t v^n=\frac{1}{\Delta t}(v^{n+1}-v^n),$$

and $t^{n+\frac{1}{2}}=(n+\frac{1}{2})\Delta t$.

Further, we need consider the numerical approximation of the integral over $[0,t]$ on the right-hand side of (2.4b). For simplicity, introduce the notation

$$I(t)=\int_o^t g(\tau)z(\tau)d\tau. \tag{2.5}$$

For avoiding the corresponding algebraic system to be complex, we introduce the extrapolation technique to treat the right-hand side term in (2.4b) at $t=t^{n+1}$:

$$I(t^{n+1})\approx 2I(t^n)-I(t^{n-1}). \tag{2.6}$$

For approximating the integral $I(t^n)$, the middle point formula is given as

$$\int_0^{t^n} g(\tau)z(\tau)d\tau\approx\sum_{l=0}^{n-1}g(t^{l+\frac{1}{2}})z^{l+1/2}\Delta t. \tag{2.7}$$

We can now give the full-discrete mixed finite element approximation to (2.3) as follows: Find $\{Q^{n+1},Z^{n+1},U^{n+1}\}\in V_h\times H_h\times V_h$ such that, for $n\geq 1$

$$(\partial_t Q^n,v_h)-(div\ Z^{n+\frac{1}{2}},v_h)=(f(t^{n+\frac{1}{2}}),v_h),\ \forall v_h\in V_h, \tag{2.8a}$$

$$\begin{aligned}(Q^{n+1},div\ \varphi_h)+(\frac{1}{b}Z^{n+1},\varphi_h)=\Big(d(t^{n+1})\Big[\sum_{l=0}^{n-1}g(t^{l+1/2})Z^{l+\frac{1}{2}}\Delta t\\ +g(t^{n-\frac{1}{2}})Z^{n-\frac{1}{2}}\Delta t\Big],\varphi_h\Big),\ \forall\varphi_h\in H_h,\end{aligned} \tag{2.8b}$$

$$(\partial_t U^n,v_h)-(Q^{n+\frac{1}{2}},v_h)=0,\ \forall v_h\in V_h, \tag{2.8c}$$

and for the first time level ($n=0$)

$$(\partial_t Q^0,v_h)-(div\ Z^{\frac{1}{2}},v_h)=(f(t^{\frac{1}{2}}),v_h),\ \forall v_h\in V_h, \tag{2.8d}$$

$$(Q^1,div\ \varphi_h)+(\frac{1}{b}Z^1,\varphi_h)=(d(t^1)g(t^1)Z^0\Delta t,\varphi_h),\forall\varphi_h\in H_h, \tag{2.8e}$$

$$(\partial_t U^0,v_h)-(Q^{\frac{1}{2}},v_h)=0,\ \forall v_h\in V_h, \tag{2.8f}$$

where U^0, Q^0 and Z^0 are the approximations of $u_0(x), q_0(x)$ and $z_0(x)$.

In order to obtain high order accuracy for the approximations at the first time level, we introduce the following modified scheme for (2.8e): for $\varphi_h \in H_h$

$$(Q^1, div\ \varphi_h) + (\frac{1}{b}Z^1, \varphi_h) = \Big(d(t^1)\big[g(t^{\frac{1}{2}})Z^{\frac{1}{2}} + (\frac{a}{b}z_0\Delta t + \frac{\partial z}{\partial t}(0)\Delta t)\big]\Delta t, \varphi_h\Big), \quad (2.8g)$$

where $\frac{\partial z}{\partial t}(0) = b\big[d(0)g(0)z_0 + \nabla f(x,0) + \nabla\big(div\ z_0\big)\big]$ is obtained from (2.2) at $t = 0$.

In the following sections, we will discuss the theoretical analysis for the semi-discrete and full-discrete mixed schemes. We assume throughout this paper that the exact solution of the equation has the smoothness required by our argument, as for ease of presentation we sometimes refrain from making the assumption explicit. For the existence, uniqueness, and regularity results of exact solutions of these type problems, the reader is referred to the papers [6] [7] [13][19] and the references quoted there.

3 Auxiliary Projection and Related Estimates

For analyzing the errors of the semi-discrete and full-discrete mixed methods (2.4) and (2.8), respectively, we will define the auxiliary projection associated with the solution of the original problem (2.2) (or (1.1)) and estimate the related errors of the projection.

For $t \in (0, T]$, define $\{\tilde{q}, \tilde{z}\}$ as the auxiliary projection of the solution $\{q, z\}$ onto $V_h \times H_h$ satisfying that

$$(q - \tilde{q}, div\ \varphi_h) + (\frac{1}{b}(z - \tilde{z}), \varphi_h) = 0, \quad \forall \varphi_h \in H_h, \quad (3.1a)$$

$$(q_t - \tilde{q}_t, v_h) - (div\ (z - \tilde{z}), v_h) = 0, \quad \forall v_h \in V_h, \quad (3.1b)$$

where the initial values $\tilde{q}(0)$ and $\tilde{z}(0)$ are approximations of $u_1(x)$ and $z_0(x)$ in V_h and H_h, respectively, and satisfy the approximation property

$$\|q(0) - \tilde{q}(0)\| + \|z(0) - \tilde{z}(0)\|_H \le Ch^{k+1}. \quad (3.2)$$

For simplicity, we can choose the initial values as the L^2-projections of $q(0) = u_1(x)$ and $z(0) = z_0(x)$ onto V_h and H_h, respectively. However, we can also define $\{\tilde{q}(0), \tilde{z}(0)\}$ as the mixed elliptic projection of $\{q(0), z(0)\}$ onto $V_h \times H_h$ by

$$(\tilde{q}(0) - q(0), div\ \varphi_h) + (\frac{1}{b}(\tilde{z}(0) - z(0)), \varphi_h) = 0, \quad \forall \varphi_h \in H_h, \quad (3.3a)$$

$$(div\ (\tilde{z}(0) - z(0)), v_h) = 0, \quad \forall v_h \in V_h, \quad (3.3b)$$

where $z(0) = z_0$, and $q(0) = u_1(x)$. All of these projections have the accuracy described in (3.2) (see, e.g. [1] [3] [14] [16]).

Let $V_h = span\{v_i(x), i = 1, 2, \cdots, L\}$, $H_h = span\{\psi_i(x), i = 1, 2, \cdots, M\}$, and $\tilde{q} = \sum_{i=1}^{L} \alpha_i(t) v_i(x)$, $\tilde{z} = \sum_{i=1}^{M} \beta_i(t)\psi_i(x)$. Let $\vec{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_L(t))^T$ and $\vec{\beta}(t) = (\beta_1(t), \beta_2(t), \cdots, \beta_M(t))^T$. Then, setting $v_h = v_i(x)(i = 1, 2, \cdots, L)$ in (3.1a) and $\phi_h = \psi_i(x)(i = 1, 2, \cdots, M)$ in (3.1b), we have the following system:

$$\mathbb{B}^T \vec{\alpha}(t) + \mathbb{C}\vec{\beta}(t) = \vec{G}(t), \qquad (3.4a)$$

$$\mathbb{A}\frac{d\vec{\alpha}(t)}{dt} - \mathbb{B}\vec{\beta}(t) = \vec{F}(t), \qquad (3.4b)$$

where $\vec{\alpha}(0)$ and $\vec{\beta}(0)$ are given from $\tilde{q}(0)$ and $\tilde{z}(0)$, respectively. In the above system, the matrices $\mathbb{A}, \mathbb{B}$, $\mathbb{C}$ and the vectors $\vec{G}$, $\vec{F}$ are in the forms

$$\mathbb{A} = \Big((v_i, v_j)\Big)_{L\times L}, \quad \mathbb{B} = \Big((div\ \psi_i, v_j)\Big)_{M\times L}, \quad \mathbb{C} = \Big((\frac{1}{b}\psi_i, \psi_j)\Big)_{M\times M}$$

and

$$\vec{G}(t) = \Big((q(t), \psi_j) + (\frac{1}{b}z, \psi_j)\Big)^T_{1\times M},$$

$$\vec{F}(t) = \Big((q_t, v_j) - (\ div\ z, v_j)\Big)^T_{1\times L}.$$

Hereafter the superscript T indicates transpose of a matrix or a vector.

Since the matrices $\mathbb{A}$ and $\mathbb{C}$ are positive definite, we know that the solution of the linear system (3.4) exists uniquely by the standard theory of ordinary differential equations. Thus, the auxiliary projection defined in (3.1) is existent and unique.

We now list the approximation results of the Raviart-Thomas-Nedelec mixed finite element space $V_h \times H_h$ ([1] [16] [18]). There exist projections:

$$\Pi_h \times P_h : H \times V \to H_h \times V_h,$$

which have the following properties:

(i) P_h is L^2-projection onto V_h and hence

$$(div\ \varphi, v - P_h v) = 0, \quad v \in V_h; \qquad (3.5a)$$

(ii) $div\Pi_h = P_h div : H \to V_h$ and thus

$$(div\ (\varphi - \Pi_h\varphi), v) = 0, \quad v \in V_h, \varphi \in H_h; \qquad (3.5b)$$

(iii) The following approximation properties hold:

$$\|\varphi - \Pi_h\varphi\| \le C\|\varphi\|_s h^s, \ 1 \le s \le k+1; \qquad (3.5c)$$

$$\|div\ (\varphi - \Pi_h\varphi)\| \le C\|div\ \varphi\|_s h^s, \ 0 \le s \le k+1; \qquad (3.5d)$$

$$\|v - P_h v\| \le C\|v\|_s h^s, \ 0 \le s \le k+1. \qquad (3.5e)$$

Here and in what follows, C will denote a generic positive constant which is independent of the parameter h (and the parameter Δt) .

We now discuss the errors of the auxiliary projection $\{\tilde{q}, \tilde{z}\}$. Let $\rho = q - \tilde{q},\ \delta = z - \tilde{z}$. Then (3.1) can be written as

$$(\rho, div\ \varphi_h) + (\frac{1}{b}\delta, \varphi_h) = 0, \ \ \forall \varphi_h \in H_h, \tag{3.6a}$$

$$(\rho_t, v_h) - (div\ \delta, v_h) = 0, \ \ \forall v_h \in V_h. \tag{3.6b}$$

Differentiating (3.6a) with respect to t, we have

$$(\rho_t, div\ \varphi_h) + (\frac{1}{b}\delta_t, \varphi_h) = 0, \ \ \forall \varphi_h \in H_h. \tag{3.7}$$

From (3.6b), (3.7), (3.5a) and (3.5b), it follows that

$$\begin{aligned}(\rho_t, \rho_t) \ &+ \ (\frac{1}{b}\delta_t, \delta) = (\rho_t, q_t - P_h q_t) + (div\ \delta, P_h q_t - \tilde{q}_t) + (\frac{1}{b}\delta_t, \delta) \\ &= \ (\rho_t, q_t - P_h q_t) + \Big(div\ [z - \Pi z + \Pi_h z - \tilde{z}], P_h q_t - \tilde{q}_t\Big) + (\frac{1}{b}\delta_t, \delta) \\ &= \ (\rho_t, q_t - P_h q_t) + (div(\Pi_h z - \tilde{z}), q_t - \tilde{q}_t) + (\frac{1}{b}\delta_t, z - \Pi_h z + \Pi_h z - \tilde{z}) \\ &= \ (\rho_t, q_t - P_h q_t) + (\frac{1}{b}\delta_t, z - \Pi_h z).\end{aligned} \tag{3.8}$$

Thus, using the Cauchy-Schwarz inequality, it can be derived from (3.8) that

$$\frac{1}{2}\|\rho_t\|^2 + (\frac{1}{b}\delta_t, \delta) \le \frac{1}{2}\|q_t - P_h q_t\|^2 + (\frac{1}{b}\delta_t, z - \Pi_h z). \tag{3.9}$$

Denote the $b-$weighted L^2-norm on H_h as

$$\|\varphi\|_b^2 = (\frac{1}{b}\varphi, \varphi), \qquad \forall \varphi \in H_h.$$

Using (3.5c) and the Cauchy-Schwarz inequality, we have the following estimate:

$$\begin{aligned}\int_0^t (\frac{1}{b}\delta_t, z - \Pi_h z)\ d\tau &= (\frac{1}{b}\delta, z - \Pi_h z)|_0^t - \int_0^t (\frac{1}{b}\delta, (z - \Pi_h z)_t) d\tau \\ &\le C\{\|\delta(0)\|_b^2 + \|z(0) - \Pi_h z(0)\|_b^2 + \|z - \Pi_h z\|_b^2 \\ &\quad + \int_0^t \|(z - \Pi_h z)_t\|_b^2 d\tau\} + \varepsilon(\int_0^t \|\delta\|_b^2 d\tau + \|\delta\|_b^2) \\ &\le C\{h^{2k+2} + \|\delta(0)\|_b^2\} + \varepsilon(\int_0^t \|\delta\|_b^2 d\tau + \|\delta\|_b^2).\end{aligned} \tag{3.10}$$

Integrating (3.9) from 0 to t and using the estimate (3.10) above, we obtain

$$\int_0^t \|\rho_t\|^2 d\tau + (1 - \varepsilon)\|\delta\|_b^2 \le C\{\|\delta(0)\|^2 + h^{2k+2}$$

$$+\int_0^T \|q_t - P_h q_t\|^2 d\tau\} + \varepsilon \int_0^t \|\delta\|_b^2 d\tau. \tag{3.11}$$

With help of (3.5e) and the Bellman inequality, from (3.11) we have

$$\int_0^t \|\rho_t\|^2 d\tau + \|\delta\|_b^2 \le C\{\|\delta(0)\|^2 + h^{2k+2}\}. \tag{3.12}$$

Noting (3.12) and the fact that

$$\|\rho\|^2 = \int_\Omega \Big(\int_0^t \rho_t d\tau + \rho(0)\Big)^2 dx \le C\int_0^t \Big(\|\rho_t\|^2 + \|\rho(0)\|^2\Big) d\tau, \qquad (3.13),$$

it can be obtained that

$$\|\rho\|^2 \le C\{\|\rho(0)\|^2 + \|\delta(0)\|^2 + h^{2k+2}\}. \tag{3.14}$$

Combining (3.12) and (3.14), we get the following approximation theorem.

Theorem 3.1 Let $\{\tilde{q},\ \tilde{z}\}$ be the projection of $\{q, z\}$ defined by (3.1a) and (3.1b). If the initial values $\{\tilde{q}(0), \tilde{z}(0)\}$ satisfies (3.2), then the following error estimate holds

$$\|q-\tilde{q}\|_{L^\infty(J;L^2)} + \|z-\tilde{z}\|_{L^\infty(J;L^2)} + \|(q-\tilde{q})_t\|_{L^2(J;L^2)} \le Ch^{k+1} \tag{3.15}$$

where C is a constant depending on $\|q\|_{L^\infty(J,H^{k+1}(\Omega))}, \|z\|_{L^\infty(J,(H^{k+1}(\Omega))^2)}$, and $\|q_t\|_{L^2(J,H^{k+1}(\Omega))}, \|z_t\|_{L^2(J,(H^{k+1}(\Omega))^2)}$.

Similarly, we have the following estimates for the time-derivatives of the auxiliary projection $\{\tilde{q}, \tilde{z}\}$ if the exact solution is sufficiently smooth:

$$\|q_t-\tilde{q}_t\|_{L^\infty(J;L^2)} + \|z_t-\tilde{z}_t\|_{L^\infty(J;L^2)} + \|(q-\tilde{q})_{tt}\|_{L^2(J;L^2)} \le Ch^{k+1}, \tag{3.16}$$

and

$$\|q_{tt}-\tilde{q}_{tt}\|_{L^\infty(J;L^2)} + \|z_{tt}-\tilde{z}_{tt}\|_{L^\infty(J;L^2)} + \|(q-\tilde{q})_{ttt}\|_{L^2(J;L^2)} \le Ch^{k+1}. \tag{3.17}$$

4 Error estimates of the semi-discrete mixed approximation

In this section we will consider the existence, uniqueness, and L^2-norm error estimates of the solution of the semi-discrete mixed finite element approximation. Let $\{\upsilon_i(x)\}_{i=1}^L$ and $\{\psi_i(x)\}_{i=1}^M$ be again the base functions of finite dimension spaces V_h and H_h, respectively. Let

$$Q = \sum_{i=1}^L \alpha_i(t)\upsilon_i(x), \quad Z = \sum_{i=1}^M \beta_i(t)\psi_i(x),\ U = \sum_{i=1}^L \gamma_i(t)\upsilon_i(x),$$

and $\vec{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_L(t))^T$, $\vec{\beta}(t) = (\beta_1(t), \beta_2(t), \cdots, \beta_M(t))^T$, and $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \cdots, \gamma_L(t))^T$. Setting $v_h = \upsilon_j(x)$, for $j = 1, 2, \cdots, L$, in (2.4a) (2.4c) and $\varphi_h = \psi_j(x)$, for $i = 1, 2, \cdots, M$, in (2.4b), we derive the ordinary system:

$$\mathbb{A}\frac{d\vec{\alpha}(t)}{dt} - \mathbb{B}\vec{\beta}(t) = \vec{F}(t), \tag{4.1a}$$

$$\mathbb{B}^T\vec{\alpha}(t) + \mathbb{C}\vec{\beta}(t) = \int_0^t \mathbb{G}(t,\tau)\vec{\beta}(\tau)\, d\tau, \tag{4.1b}$$

$$\frac{d\vec{\gamma}(t)}{dt} = \vec{\alpha}(t), \tag{4.1c}$$

with $\vec{\alpha}(0)$, $\vec{\beta}(0)$, and $\vec{\gamma}(0)$ are given from $Q(0)$, $Z(0)$, and $U(0)$, respectively. $Q(0)$, $Z(0)$, and $U(0)$ are the approximations of $u_1(x), z_0(x)$, and $u_0(x)$ onto V_h, H_h, and V_h. In the system, the matrices $\mathbb{A}, \mathbb{B}, \mathbb{C}$, and $\mathbb{G}$ and the vector $\vec{F}(t)$ are defined in the forms:

$$\mathbb{A} = \Big((v_i, v_j)\Big)_{L\times L}, \quad \mathbb{B} = \Big((div\ \psi_i, v_j)\Big)_{M\times L},$$

$$\mathbb{C} = \Big((\frac{1}{b}\psi_i, \psi_j)\Big)_{M\times M}, \quad \mathbb{G}(t,\tau) = \Big((d(t)g(\tau)\psi_i, \psi_j)\Big)_{M\times M},$$

and

$$\vec{F}(t) = \Big((f(t), v_j)\Big)^T_{1\times L}.$$

Differentiating (4.1b) with respect to t, combining (4.1a) and using the property that $\mathbb{A}$ and $\mathbb{C}$ are positive-definite, it can be derived that

$$\begin{aligned} \mathbb{C}\frac{d\vec{\beta}(t)}{dt} &= -\mathbb{B}^T\mathbb{A}^{-1}\mathbb{B}\vec{\beta}(t) + \mathbb{G}(t,\tau)\vec{\beta}(t) \\ &\quad + \int_0^t \mathbb{G}_t(t,\tau)\vec{\beta}(\tau)\, d\tau - \mathbb{B}^T\mathbb{A}^{-1}\vec{F}(t), \end{aligned} \tag{4.2}$$

where

$$\mathbb{G}_t(t,\tau) = \Big((d_t(t)g(\tau)\psi_i, \psi_j)\Big)_{M\times M}.$$

By the regularity theory of ordinary integro-differential equations (see [2]), there exists a unique solution of the system (4.1a), (4.2) and (4.1c). Thus, we have the following existence and uniqueness theorem.

Theorem 4.1 There exists a unique solution $\{Q(t), Z(t), U(t)\} \in V_h \times H_h \times V_h$ for the semi-discrete mixed finite element scheme (2.4) on $(0, T]$.

Now we estimate the errors of the semi-discrete mixed scheme. Let $\theta = \tilde{q} - Q$, $\pi = \tilde{z} - Z$ and let $\rho = q - \tilde{q}, \delta = z - \tilde{z}$ be the same as in Section 3. Then $q - Q = \rho + \theta$, and $z - \tilde{z} = \delta + \pi$.

Subtracting (2.4) from (2.3) and using the definition of the auxiliary projection (3.1), we have the error equations:

$$(\theta_t, v) - (div\ \pi, v_h) = 0, \quad \forall v_h \in V_h; \tag{4.3a}$$

$$(\theta, div\ \varphi_h) + (\frac{1}{b}\pi, \varphi_h) = \Big(d(t)\int_0^t g(\delta+\pi)\, d\tau, \varphi_h\Big), \quad \forall\varphi_h \in H_h; \tag{4.3b}$$

$$(u_t - U_t, v_h) = (\rho + \theta, v_h); \quad \forall v_h \in V_h. \tag{4.3c}$$

Differentiating (4.3b) with respect to t leads to the result that for $\varphi_h \in H_h$,

$$(\theta_t, div\ \varphi_h) + (\frac{1}{b}\pi_t, \varphi_h) = -\Big(\frac{a}{b}d\int_0^t e^{\frac{a}{b}\tau}(\delta+\pi)\ d\tau, \varphi_h\Big) + \Big(\frac{a}{b^2}(\delta+\pi), \varphi_h\Big). \quad (4.4)$$

Combining (4.3a) with $v_h = \theta_t$ and (4.4) with $\varphi_h = \pi$, it holds that

$$\|\theta_t\|^2 + (\frac{1}{b}\pi_t, \pi) = \Big(\frac{a}{b^2}(\delta+\pi), \pi\Big) - \Big(\frac{a}{b}d\int_0^t g(\delta+\pi)d\tau, \pi\Big). \quad (4.5)$$

Integrating the equation (4.5) from 0 to $t \in J$ and using the $b-$weighted norm $\|\cdot\|_b$, we get

$$\int_0^t \|\theta_t(\tau)\|^2 d\tau + \|\pi\|_b^2(t) \le C\{\|\pi(0)\|_b^2 + \int_0^t (\|\pi\|_b^2 + \|\delta\|^2)\ d\tau\}. \quad (4.6)$$

Applying the Bellman inequality to (4.6) and using the results in Theorem 3.1, we obtain

$$\int_0^t \|\theta_t\|^2 d\tau + \|\pi\|_b^2(t) \ \le\ C\{\|\pi(0)\|_b^2 + h^{2k+2}\}. \quad (4.7)$$

Now choosing $v_h = \theta$ in (4.3a) and $\varphi_h = \pi$ in (4.3b) and summing up these two equations lead to the error relation

$$(\theta_t, \theta) + (\frac{1}{b}\pi, \pi) = \Big(d\int_0^t g(\delta+\pi)d\tau, \pi\Big). \quad (4.8)$$

Further, arguing similarly as in deriving (4.7) we get

$$\|\theta\|^2(t) \ \le\ C\{\|\theta(0)\|^2 + \|\pi(0)\|^2 + h^{2k+2}\}. \quad (4.9)$$

Finally, from (4.3c) we have the error equation:

$$(u_t - U_t, u - U) = (u_t - U_t, u - P_h u) + (\rho + \theta, P_h u - U). \quad (4.10)$$

Integrating (4.10) from 0 to t and using integration by parts with respect to τ for $\int_0^t (u_t - U_t, u - P_h u)\ d\tau$, it holds that

$$\begin{aligned} \frac{1}{4}\|u-U\|^2(t) \ &\le\ \frac{3}{2}\|u-U\|^2(0) + \frac{1}{2}\|u - P_h u\|^2(0) + 2\|u - P_h u\|^2 \\ &+\ \int_0^t \Big(\|u_t - P_h u_t\|^2 + \|u - P_h u\|^2 + \|\rho\|^2 + \|\theta\|^2\Big)\ d\tau \\ &+\ \int_0^t \|u-U\|^2\ d\tau \end{aligned} \quad (4.11)$$

Thus, using (4.9) and (3.5e), we obtain from (4.11) that

$$\|u-U\|^2 \le C\{\|u(0) - U(0)\|^2 + \|\theta(0)\|^2 + h^{2k+2}\}. \quad (4.12)$$

If now the initial approximations $U(0), Q(0)$, and $Z(0)$ satisfy the property

$$\|u_0 - U(0)\| + \|u_1 - Q(0)\| + \|z_0 - Z(0)\| \ \le\ Ch^{k+1}, \quad (4.13)$$

then combination of (4.7), (4.9) and (4.12) leads to the following error estimate theorem.

Theorem 4.2 Let $\{u, q, z\}$ be the solution of (2.2) (or (1.1)) and let $\{U, Q, Z\}$ be the solution of the semi-discrete mixed finite element method (2.4). If $u, q \in L^\infty(0, T; H^{k+1}(\Omega)) \cap H^1(0, T; H^{k+1}(\Omega))$, $z \in L^\infty(0, T; (H^{k+1}(\Omega))^2) \cap H^1(0, T; (H^{k+1}(\Omega))^2)$, and $\{U(0), Q(0), Z(0)\}$ satisfies (4.13), then there exits a positive constant C independent of h such that

$$\|u - U\|_{L^\infty(J;L^2)} + \|q - Q\|_{L^\infty(J;L^2)} + \|z - Z\|_{L^\infty(J;L^2)} \le Ch^{k+1}. \tag{4.14}$$

As a remark, it is clear that if choosing the initial values $\{Q(0), Z(0)\}$ as the mixed projection $\{\tilde{q}, \tilde{z}\}$ defined in (3.3) and choosing $U(0)$ as the L^2-projection of $u(0)$ onto V_h then the initial approximation property (4.13) holds (see, e.g. [3][16][17]).

5 Error estimates of the full-discrete mixed approximation

In this section we shall discuss the full-discrete mixed finite element approximations. In our full-discrete scheme, the right-hand side term in the scheme (2.8) was treated by using the extrapolation technique. The scheme is much effective in computation and have the second order truncation error in the time step size. The current level unknown variables are only come from the left-hand side of the system. The saddle-point type coefficient matrix is similar to that obtained from the parabolic problem (see [9] [17]). Thus, it is easy to see the existence and uniqueness of the solution of the system (2.8). Here, we shall mainly show the error estimate theorem.

Theorem 5.1 Let $\{u, q, z\}$ be the exact solution of the original problem (2.2) (or (1.1) and let $\{U^n, Q^n, z^n\}$ be the solution of the full-discrete mixed finite element scheme (2.8). If choosing the initial values $U^0 = \tilde{u}(0)$, $Q^0 = \tilde{q}(0)$ and $Z^0 = \tilde{z}(0)$ as defined in (3.3) and if the first time level values U^1, Q^1 and Z^1 are defined as in (2.8d), (2.8f) and (2.8g), then there exists a constant $C > 0$ independent of step sizes h and Δt such that

$$\max_{0\le n\le N}\{\|u^n - U^n\|^2 + \|q^n - Q^n\|^2 + \|z^n - Z^n\|^2\} \le C\{(\Delta t)^4 + h^{2k+2}\}. \tag{5.1}$$

Proof. Let

$$u^n - U^n = \xi^n,\ q^n - Q^n = q^n - \tilde{q}^n + \tilde{q}^n - Q^n = \rho^n + \theta^n,$$

$$z^n - Z^n = z^n - \tilde{z}^n + \tilde{z}^n - Z^n = \delta^n + \pi^n.$$

Then $\theta^0 = 0$ and $\pi^0 = 0$. Setting $t = t^{n+\frac{1}{2}}$ in (2.3a) and (2.3c) and $t = t^{n+1}$ in (2.3b), we have that for $n \ge 0$,

$$(q_t(t^{n+\frac{1}{2}}), v_h) - (div\ z(t^{n+\frac{1}{2}}), v_h) = (f(t^{n+\frac{1}{2}}), v_h) \quad v_h \in V_h, \tag{5.2a}$$

$$(q^{n+1}, div\ \varphi_h) + (\frac{1}{b}z^{n+1}, \varphi_h) = (d(t^{n+1})I(t^{n+1}), \varphi_h), \quad \varphi_h \in H_h, \qquad (5.2b)$$

$$(u_t(t^{n+\frac{1}{2}}), v_h) = (q(t^{n+\frac{1}{2}}), v_h), \quad v_h \in V_h. \qquad (5.2c)$$

Subtracting every equation in (2.8) from the corresponding one in (5.2) and using the definition of $\{\tilde{q},\ \tilde{z}\}$ in (3.1) yield the following error equations: for $n \geq 0$,

$$(\partial_t\theta^n, v_h) - (div\ \pi^{n+\frac{1}{2}}, v_h) = (\sum_{i=1}^{2} T_i^n, v_h), \quad \forall v_h \in V_h, \qquad (5.3a)$$

$$(\theta^{n+1}, div\ \varphi) + (\frac{1}{b}\pi^{n+1}, \varphi_h) = (d(t^{n+1})\sum_{i=1}^{4} G_i^n, \varphi - h), \quad \forall \varphi_h \in H_h, \qquad (5.3b)$$

$$\begin{aligned}(\partial_t\xi^n, v_h) &= (\rho^{n+\frac{1}{2}} + \theta^{n+\frac{1}{2}}, v_h) + (\partial_t u^n - u_t(t^{n+\frac{1}{2}}), v_h)\\ &\quad + (q(t^{n+\frac{1}{2}}) - q^{n+\frac{1}{2}}, v_h),\ \forall v_h \in V_h, \qquad (5.3c)\end{aligned}$$

where for $n \geq 0$,

$$T_1^n = \partial_t\tilde{q}^n - \tilde{q}_t(t^{n+\frac{1}{2}}), \quad T_2^n = \tilde{z}(t^{n+\frac{1}{2}}) - \tilde{z}^{n+\frac{1}{2}}, \qquad (5.4a)$$

and for $n \geq 1$,

$$\begin{aligned}
G_1^n &= I(t^{n+1}) - \big(2I(t^n) - I(t^{n-1})\big),\\
G_2^n &= 2\Big(I(t^n) - \sum_{l=0}^{n-1} g(t^{l+\frac{1}{2}})z^{l+\frac{1}{2}}\Delta t\Big)\\
&\quad - \Big(I(t^{n-1}) - \sum_{l=0}^{n-2} g(t^{l+\frac{1}{2}})z^{l+\frac{1}{2}}\Delta t)\Big),\\
G_3^n &= 2\sum_{l=0}^{n-1} g(t^{l+\frac{1}{2}})\delta^{l+\frac{1}{2}}\Delta t - \sum_{l=0}^{n-2} g(t^{l+\frac{1}{2}})\delta^{l+\frac{1}{2}}\Delta t,\\
G_4^n &= 2\sum_{l=0}^{n-1} g(t^{l+\frac{1}{2}})\pi^{l+\frac{1}{2}}\Delta t - \sum_{l=0}^{n-2} g(t^{l+\frac{1}{2}})\pi^{l+\frac{1}{2}}\Delta t,
\end{aligned} \qquad (5.4b)$$

and for $n = 0$,

$$\begin{aligned}
G_1^0 &= \frac{a}{b}z_0(\Delta t)^2 + \frac{\partial z}{\partial t}(0)(\Delta t)^2,\\
G_2^0 &= I(t^1) - g(t^{\frac{1}{2}})z^{\frac{1}{2}}\Delta t,\\
G_3^0 &= g(t^{\frac{1}{2}})\delta^{\frac{1}{2}}\Delta t,\\
G_4^0 &= g(t^{\frac{1}{2}})\pi^{\frac{1}{2}}\Delta t.
\end{aligned} \qquad (5.4c)$$

Here we have used the fact that $I(t^0) = 0$ and the definition that, for any function v, $\sum_{l=0}^{n-2} g(t^{l+\frac{1}{2}})v^{l+\frac{1}{2}} = 0$ if $n = 1$.

Now, we estimate the bounds of θ^n and π^n for $n \geq 2$. Subtracting (5.3b) at $n-1$ level from itself at n level, we have that for $n \geq 1$,

$$(\theta^{n+1}-\theta^n, div\ \varphi_h) + (\frac{1}{b}(\pi^{n+1}-\pi^n), \varphi_h) = \Big(\big[d(t^{n+1})-d(t^n)\big]\sum_{i=1}^{4} G_i^n$$

$$+d(t^n)\sum_{i=1}^{4}(G_i^n - G_i^{n-1}), \varphi_h), \quad \forall \varphi_h \in H_h. \tag{5.5}$$

Choosing $v_h = \theta^{n+1}-\theta^n$ in (5.3a) and $\varphi_h = \pi^{n+\frac{1}{2}}$ in (5.5), from these two equations we have

$$(\partial_t\theta^n, \theta^{n+1}-\theta^n) + (\frac{1}{b}(\pi^{n+1}-\pi^n), \pi^{n+\frac{1}{2}}) = (\sum_{i=1}^{2} T_i^n, \theta^{n+1}-\theta^n)$$

$$+\sum_{i=1}^{4}\Big(\frac{d(t^{n+1})-d(t^n)}{\Delta t}G_i^n + d(t^n)\frac{G_i^n - G_i^{n-1}}{\Delta t}, \pi^{n+\frac{1}{2}}\Big)\Delta t. \tag{5.6}$$

Noting that

$$\Big(\frac{1}{b}(\pi^{n+1}-\pi^n), \frac{\pi^{n+1}+\pi^n}{2}\Big) = \frac{1}{2}(\|\pi^{n+1}\|_b^2 - \|\pi^n\|_b^2)$$

and using the Cauchy-Schwarz inequality, we obtain that for $n \geq 1$

$$\|\partial_t\theta^n\|^2\Delta t + \frac{1}{2}(\|\pi^{n+1}\|_b^2 - \|\pi^n\|_b^2) \leq \frac{1}{2}\|\partial_t\theta^n\|^2\Delta t + \frac{1}{4}\|\pi^{n+\frac{1}{2}}\|_b^2\Delta t$$

$$+\sum_{i=1}^{2}\|T_i^n\|^2\Delta t + C\sum_{i=1}^{4}\Big\{\|G_i^n\|^2 + \Big\|\frac{G_i^n - G_i^{n-1}}{\Delta t}\Big\|^2\Big\}\Delta t. \tag{5.7}$$

We now need estimate every term on the right-hand side of the above inequality. Noting (4.3a), (5.8), (5.9) and the fact that for any function $p(t)$

$$\frac{p^{n+1}+p^n}{2} = p(t^{n+\frac{1}{2}}) + \frac{(\Delta t)^2}{8}\frac{\partial^2 p}{\partial t^2}(t^{n+\frac{1}{2}}) + O((\Delta t)^3), \tag{5.8}$$

$$\frac{p^{n+1}-p^n}{\Delta t} = \frac{\partial p}{\partial t}(t^{n+\frac{1}{2}}) - \frac{(\Delta t)^2}{24}\frac{\partial^3 p}{\partial t^3}(t^{n+\frac{1}{2}}) + O((\Delta t)^3), \tag{5.9}$$

it is easy to derive that for $n \geq 0$,

$$\|T_1^n\|^2 + \|T_2^n\|^2 \leq C(\Delta t)^4. \tag{5.10}$$

The extrapolation error is

$$I(t^{n+1}) = 2I(t^n) - I(t^{n-1}) + \frac{\partial^2 I}{\partial t^2}(t^{n+1})(\Delta t)^2 + O((\Delta t)^3), \tag{5.11}$$

so we have

$$\|G_1^n\|^2 = \|I(t^{n+1}) - 2I(t^n) + I(t^{n-1})\|^2 \leq C(\Delta t)^4 \tag{5.12}$$

for $n \geq 1$. From the error of the composite middle point rule, we have that for smooth functions g and z,

$$I(t^n) = \sum_{l=0}^{n-1} g(t^{l+\frac{1}{2}}) z^{l+\frac{1}{2}} \Delta t + O((\Delta t)^2).$$

Thus we have the following estimate: for $n \geq 1$

$$\begin{aligned} \|G_2^n\|^2 &= \Big\|2\Big(I(t^n) - \sum_{l=0}^{n-1} g(t^{l+\frac{1}{2}}) z^{l+\frac{1}{2}} \Delta t\Big) - \Big(I(t^{n-1}) + \sum_{l=0}^{n-2} g(t^{l+\frac{1}{2}}) z^{l+\frac{1}{2}} \Delta t)\Big)\Big\|^2 \\ &\leq C(\Delta t)^4. \end{aligned} \tag{5.13}$$

For estimating G_3^n and G_4^n, use the Cauchy-Schwarz inequality to obtain that

$$\begin{aligned} \|G_3^n\|^2 &= \Big\| \sum_{l=0}^{n-1} g(t^{l+\frac{1}{2}}) \delta^{l+\frac{1}{2}} \Delta t + g(t^{n-\frac{1}{2}}) \delta^{n-\frac{1}{2}} \Delta t \Big\|^2 \\ &\leq C \sum_{l=0}^{n} \|\delta^l\|^2 \Delta t, \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \|G_4^n\|^2 &= \| \sum_{l=0}^{n-1} g(t^{l+\frac{1}{2}}) \pi^{l+\frac{1}{2}} \Delta t + g(t^{n-\frac{1}{2}}) \pi^{n-\frac{1}{2}} \Delta t \|^2 \\ &\leq C \sum_{l=0}^{n} \|\pi^l\|_b^2 \Delta t \end{aligned} \tag{5.15}$$

for $n \geq 1$. We now consider the estimation of $G_i^n - G_i^{n-1}$ for $n \geq 2$. From the definition of G_1^n in (5.4b) and using the error relation (5.11), we have

$$\begin{aligned} \Big\| \frac{G_1^n - G_1^{n-1}}{\Delta t} \Big\|^2 &= \Big\| \frac{1}{\Delta t} \Big((I(t^{n+1}) - 2I(t^n) + I(t^{n-1})) \\ &\quad - (I(t^n) - 2I(t^{n-1}) + I(t^{n-2})) \Big) \Big\|^2 \leq C(\Delta t)^4 \end{aligned} \tag{5.16}$$

for $n \geq 2$. Similarly, we have

$$\begin{aligned} \Big\| \frac{G_2^n - G_2^{n-1}}{\Delta t} \Big\|^2 &= \Big\| \frac{1}{\Delta t} \Big(2 \int_{t^{n-1}}^{t^n} g(\tau) z(\tau) d\tau - 2g(t^{n-\frac{1}{2}}) z^{n-\frac{1}{2}} \Delta t) \\ &\quad - \int_{t^{n-2}}^{t^{n-1}} g(\tau) z(\tau) d\tau + g(t^{n-\frac{3}{2}}) z^{n-\frac{3}{2}} \Delta t) \Big) \Big\|^2 \\ &\leq C(\Delta t)^4, \end{aligned} \tag{5.17}$$

$$\begin{aligned} \Big\| \frac{G_3^n - G_3^{n-1}}{\Delta t} \Big\|^2 &= \|2g(t^{n-\frac{1}{2}}) \delta^{n-\frac{1}{2}} - g(t^{n-\frac{3}{2}}) \delta^{n-\frac{3}{2}}\|^2 \\ &\leq C\{\|\delta^{n-\frac{1}{2}}\|^2 + \|\delta^{n-\frac{3}{2}}\|^2\}, \end{aligned} \tag{5.18}$$

and

$$\Big\|\frac{G_4^n - G_4^{n-1}}{\Delta t}\Big\|^2 = \|2g(t^{n-\frac{1}{2}})\pi^{n-\frac{1}{2}} - g(t^{n-\frac{3}{2}})\pi^{n-\frac{3}{2}}\|^2 \leq C\{\|\pi^{n-\frac{1}{2}}\|_b^2 + \|\pi^{n-\frac{3}{2}}\|_b^2\} \tag{5.19}$$

for $n \geq 2$.

We now estimate$\|\frac{G_i^1 - G_i^0}{\Delta t}\|^2$ for $i = 1, 2, 3, 4$. Noting the definition of G_i^0 in (5.4c), and (5.11) and the length of the first interval $[0, t^1]$ is Δt, it holds that

$$\begin{aligned}
\Big\|\frac{G_1^1 - G_1^0}{\Delta t}\Big\|^2 &= \Big\|\frac{1}{\Delta t}\big(I(t^2) - 2I(t^1) - (\frac{a}{b}z_0 + \frac{\partial z}{\partial t}(0))(\Delta t)^2\big)\Big\|^2 \\
&\leq C(\Delta t)^4, \\
\Big\|\frac{G_2^1 - G_2^0}{\Delta t}\Big\|^2 &= \Big\|\frac{1}{\Delta t}\Big(I(t^1) - g(t^{\frac{1}{2}})z^{\frac{1}{2}}\Delta t\Big)\Big\|^2 \leq C(\Delta t)^4, \\
\|\frac{G_3^1 - G_3^0}{\Delta t}\|^2 &= \Big\|g(t^{\frac{1}{2}})\delta^{\frac{1}{2}}\Big\|^2 \leq C\|\delta^{\frac{1}{2}}\|^2, \\
\Big\|\frac{G_4^1 - G_4^0}{\Delta t}\Big\|^2 &= \|g(t^{\frac{1}{2}})\pi^{\frac{1}{2}}\|^2 \leq C\|\pi^{\frac{1}{2}}\|^2.
\end{aligned} \tag{5.20}$$

Thus, from (5.7) and using the estimates (5.10)-(5.20), we have

$$\begin{aligned}
\|\partial_t\theta^n\|^2\Delta t + \|\pi^{n+1}\|_b^2 \leq\ & \|\pi^n\|_b^2 + \frac{1}{2}\|\pi^{n+\frac{1}{2}}\|_b^2\Delta t \\
& + C\big\{(\Delta t)^4 + \sum_{l=0}^{n}(\|\delta^l\|^2 + \|\pi^l\|_b^2)\Delta t\big\}\Delta t.
\end{aligned} \tag{5.21}$$

Summing (5.21) with respect to n from 1 to $m-1$, we obtain

$$\sum_{n=1}^{m-1}\|\partial_t\theta^n\|^2\Delta t + \|\pi^m\|_b^2 \leq \|\pi^1\|_b^2 + C\big\{(\Delta t)^4 + \sum_{n=1}^{m}\big(\|\delta^n\|^2 + \|\pi^n\|_b^2\big)\Delta t\big\} \tag{5.22}$$

for $2 \leq m \leq N$. Applying Gronwall's lemma and Theorem 3.1, from (5.22) we get the following estimate:

$$\sum_{n=1}^{m-1}\|\partial_t\theta^n\|^2\Delta t + \|\pi^m\|_b^2 \leq C\{\|\pi^1\|_b^2 + (\Delta t)^4 + h^{2k+2}\} \tag{5.23}$$

for $2 \leq m \leq N$. Setting $v_h = \theta^{n+1}$ in (5.3a) and $\varphi_h = \pi^{n+\frac{1}{2}}$ in (5.3b) and adding these two equations, we have

$$(\partial_t\theta^n, \theta^{n+1})) + (\frac{1}{b}\pi^{n+1}, \pi^{n+\frac{1}{2}}) = \Big(\sum_{i=1}^{2}T_i^n, \theta^{n+1}\Big) + \Big(d(t^{n+1})\sum_{i=1}^{4}G_i^n, \pi^{n+\frac{1}{2}}\Big). \tag{5.24}$$

Noting the relations:

$$(\partial_t\theta^n, \theta^{n+1}) = \frac{1}{2\Delta t}\Big(\|\theta^{n+1}\|^2 - \|\theta^n\|^2\Big) + \frac{1}{2\Delta t}(\theta^{n+1} - \theta^n, \theta^{n+1} - \theta^n)$$

and

$$(\frac{1}{b}\pi^{n+1},\pi^{n+\frac{1}{2}}) = \frac{1}{4}\Big(\|\pi^{n+1}\|_b^2 - \|\pi^n\|_b^2\Big) + \frac{1}{4}\Big(\frac{1}{b}\pi^{n+1}+\pi^n,\pi^{n+1}+\pi^n\Big),$$

we derive on multiplying (5.24) with Δt and using the Cauchy-Schwarz inequality that

$$\frac{1}{2}\Big(\|\theta^{n+1}\|^2 - \|\theta^n\|^2\Big) + \frac{\Delta t}{4}(\|\pi^{n+1}\|_b^2 - \|\pi^n\|_b^2)$$

$$\leq \|\theta^{n+1}\|^2\Delta t + \sum_{i=1}^{2}\|T_i^n\|^2\Delta t + C\{\sum_{i=1}^{4}\|G_i^n\|^2 + \|\pi^{n+\frac{1}{2}}\|_b^2\}\Delta t. \tag{5.25}$$

Taking the sum of the equation (5.25) with respect to n from 1 to $m-1$ and using the estimates of T_i^n and G_i^n, we get that for $m \geq 2$

$$\|\theta^m\|^2 \leq \|\theta^1\|^2 + \sum_{n=0}^{m}\|\theta^n\|^2\Delta t + C\{(\Delta t)^4 + \sum_{n=0}^{m}(\|\delta^n\|^2 + \|\pi^n\|^2)\Delta t\}. \tag{5.26}$$

Using (5.23) and Theorem 3.1 in (5.26) and then applying the Gronwall lemma to the inequality thus obtained we arrive at

$$\|\theta^m\|^2 \leq C\{\|\theta^1\|^2 + \|\pi^1\|_b^2 + (\Delta t)^4 + h^{2k+2}\}. \tag{5.27}$$

Finally, we estimate the errors $\|\theta^1\|$ and $\|\pi^1\|_b$ at the first time step. Taking $v_h = \theta^1$ in (4.2a), $\varphi_h = \pi^{\frac{1}{2}}$ in (4.2b), and adding these two equations lead to

$$(\partial_t\theta^0,\theta^1) + (b^{-1}\pi^1,\pi^{\frac{1}{2}}) = (T_1^0 + T_2^0,\theta^1) + (d(t^1)\sum_{i=1}^{4}G_i^0,\pi^{\frac{1}{2}}). \tag{5.28}$$

Since $\theta^0 = 0$ and $\pi^0 = 0$ then (5.28) can be rewritten as

$$(\partial_t\theta^0,\theta^1-\theta^0) + \frac{1}{2}(\frac{1}{b}\pi^1,\pi^1) = (T_1^0 + T_2^0,\theta^1-\theta^0) + \frac{1}{2}(d(t^1)\sum_{i=1}^{4}G_i^0,\pi^1). \tag{5.29}$$

By the Cauchy-Schwarz inequality, we see that

$$\|\partial_t\theta^0\|^2\Delta t + \|\pi^1\|_b^2 \leq C\{\sum_{i=1}^{2}\|T_i^0\|^2\Delta t + \sum_{i=1}^{4}\|G_i^0\|^2\}. \tag{5.30}$$

From the definitions of G_i^0 in (5.4c), we have

$$\|G_1^0\|^2 = \|\big(\frac{a}{b}z_0 + \frac{\partial z}{\partial}(0)\big)(\Delta t)^2\|^2 \leq C(\Delta t)^4, \tag{5.31}$$

$$\|G_2^0\|^2 = \|\int_0^{t^1} e^{\frac{a}{b}\tau}z(\tau)d\tau - g(t^{\frac{1}{2}})z^{\frac{1}{2}}\Delta t\|^2 \leq C(\Delta t)^4, \tag{5.32}$$

$$\|G_3^0\|^2 = \|g(t^{\frac{1}{2}})\delta^{\frac{1}{2}}\Delta t\|^2 \le C\|\delta^{\frac{1}{2}}\|^2(\Delta t)^2, \tag{5.33}$$

and

$$\|G_4^0\|^2 = \|g(t^{\frac{1}{2}})\pi^{\frac{1}{2}}\Delta t\|^2 \le C\|\pi^{\frac{1}{2}}\|^2(\Delta t)^2. \tag{5.34}$$

Thus, from (5.30) and using the estimates (5.31)-(5.34) and (5.10), it can be derived that

$$\|\partial_t\theta^0\|^2\Delta t + \|\pi^1\|_b^2 \le C\{(\Delta t)^4 + h^{2k+2}\}. \tag{5.35}$$

Further, from (5.35) and the fact that $\theta^0 = 0$ it is clear to see that

$$\|\theta^1\|^2 = \|\partial_t\theta^0\|^2(\Delta t)^2 \le C\{(\Delta t)^4 + h^{2k+2}\}. \tag{5.36}$$

Thus, combining (5.23), (5.27), (5.35) and (5.36), we obtain the final estimate of θ^m and π^m: for $0 \le m \le N$

$$\|\theta^m\|^2 + \|\pi^m\|_b^2 \le C\{(\Delta t)^4 + h^{2k+2}\}. \tag{5.37}$$

On the other hand, setting $v = \xi^{n+1} + \xi^n$ in (5.3c) we get

$$\|\xi^{n+1}\|^2 \le \|\xi^n\|^2 + \frac{1}{2}\Delta t\|\xi^{n+1} + \xi^n\|^2 + 2\{\|\rho^{n+\frac{1}{2}}\|^2 + \|\theta^{n+\frac{1}{2}}\|^2$$

$$+\|q(t^{n+\frac{1}{2}}) - q^{n+\frac{1}{2}}\|^2 + \|\partial_t u^n - u_t(t^{n+\frac{1}{2}})\|\}\Delta t. \tag{5.38}$$

Similarly, we have that for $0 \le m \le N$

$$\|\xi^m\|^2 \le C\big\{(\Delta t)^4 + h^{2k+2}\big\}. \tag{5.39}$$

This ends the proof of the theorem.

Acknowledgements.

The work of D. Liang was supported by the Natural Sciences and Engineering Research Council, Canada. He also thanks the School of Mathematical and Information Sciences at Coventry University, UK for the support during his visit to the School.

References

[1] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer Series in Computational Mathematics Vol. 15, Springer, New York, 1991.

[2] T.A. Burton, Volterra Integral and Differential Equations, Academic Press, 1983.

[3] P. G. Ciarlet, The Finite Element Methods for Elliptic Problems, North Holland, 1978.

[4] L.C. Cowsar, T.F.Dupont and M.F. Wheeler, A priori estimates for mixed finite element methods for the wave equation, *Comput. Methods Appl. Mech. Engrg.* **82** (1990), 205-222.

[5] L. Cowsar, T. Dupont and M. F. Wheeler, A priori estimates for mixed finite element approximations of second-order hyperbolic equations with absorbing boundary conditions, *SIAM J. Numer. Anal.* **33** (1996), 492-504.

[6] W.E. Fitzgibbon, Strongly damped quasilinear evolution equations, *J. Math. Anal. Appl.* **79** (1981), 536-550.

[7] E. Greenwell Yanik and G. Fairweather, Finite element methods for nonlinear parabolic integro-differential equations, *Nonlinear Analysis TMA* **12** (1988), 785-809.

[8] T. Geveci, On the application of mixed finite element methods to the wave equation, *Math. Model. Numer. Anal.* **22** (1988), 243–250.

[9] C. Johson and V.Thomée, Error estimates for some mixed finite element methods for parabolic type problems, *RAIRO Anal. Numer.* **15** (1981), 41-78.

[10] S. Larsson, V. Thomée, and L.B. Wahlbin, Finite element methods for a strongly damped wave equation, *IMA J. Numer. Anal.* **11** (1991), 115-142.

[11] Y. Lin, V. Thomée, and L.B. Wahlbin, Ritz-Volterra projections to finite element spaces and applications to integrodifferential and related equations, *SIAM J. Numer. Anal.* **28** (1991), 1047-107.

[12] C.G. Makridakis, On mixed finite element methods for linear elastodynamics, *Numer. Math.* **61** (1992), 235-260.

[13] P. Massatt, Limiting behavior for strongly damped nonlinear wave equations, *J. Differential Equations* **48** (1983), 334-349.

[14] J.C. Nedelec, Mixed finite elements in $\mathbb{R}^3$, *Numer. Math.* **35** (1980), 315-341.

[15] A.K. Pani and J. Yuan, Mixed Finite Element Method for a Strongly Damped Wave Equation, *Numer Methods Partial Differential Eqns* **17** (2001), 105-119.

[16] R. A. Raviart and J. M. Thomas, A mixed finite element method for second-order elliptic problems, *Lecture Notes in Math.* **606**, Springer, New York, 1977, pp. 292–315.

[17] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer-Verlag, Berlin, 1997.

[18] V. Thomée and N.-Y. Zhang, Error estimates for semi-discrete finite element methods for parabolic integro-differential equations, *Math. Comp.* **53** (1989), 121-139.

[19] G.F. Webb, Existence and asymptotic behavior for a strongly damped nonlinear wave equation, *Can. J. Math.* **32** (1980), 631-643.