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Commutator estimates, Besov spaces and scattering problems for the acoustic wave propagation in perturbed stratified fluids

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Abstract

Mourre's commutator method is used together with Besov spaces, in this paper, to settle the question of uniqueness of solutions to the scattering problem for the acoustic wave propagation in perturbed stratified fluids in $\mathbf{R}^n$ with $n \geqslant 2$. The uniqueness of solutions is established by introducing a radiation condition in the framework of Besov spaces, and is then used to prove resolvent estimates in the Besov space setting for the acoustic propagator in perturbed stratified fluids. These estimates are 'sharp' in a sense made precise in the text and are important in establishing existence of solutions to the scattering problem.

1. *Introduction and main results*

The spectral and scattering theory for acoustic propagators $H = -C^2(z)\Delta$ in perturbed stratified fluids have been studied recently by several authors [**2**, **3**, **5**–**7**, **12**, **13**, **19**–**24**]. Under suitable assumptions on the behaviour of sound speed $C(z)$ at infinity, the absence of eigenvalues and the principles of limiting absorption and limiting amplitude have been proved and the scattering theory has also been developed. What we are going to do in the present paper is to introduce a radiation condition for such propagators and to study the uniqueness of solutions to the steady-state scattering problem for the acoustic wave propagation in perturbed stratified fluids under this radiation condition. For simplicity and to fix ideas, we consider, in the present paper, the simple case of the Pekeris profile $C_0(y)$ (see (1·2) below). However, we should remark that the method also works for more general situations, in particular multidimensional ones with non-constant density $\rho_0(y)$ as those considered in [**5**, **6**]. To formulate the results obtained in this paper precisely, several assumptions and notations are required.

We work in the $n+1$-dimensional space $\mathbf{R}^{n+1}$ with $n \geqslant 1$ and write the coordinates in $\mathbf{R}^{n+1}$ as $z = (x, y)$ with $y \in \mathbf{R}$ and $x = (x_1, ..., x_n) \in \mathbf{R}^n$. We also write $r = |z|$. Let Δ be the Laplacian in $\mathbf{R}^{n+1}$ and let $C_0(y) > 0$ be the sound speed in the fluid under consideration, which depends only on the depth variable y. Then the acoustic wave in the stratified fluid is governed by the wave equation

$$\frac{\partial^2 w}{\partial t^2} - C_0^2(y)\Delta w = 0, \tag{1·1}$$

where $w(t,x,y)$ (the excess pressure or the acoustic potential) is a real-valued function defined for $t\in\mathbf{R}$, $x\in\mathbf{R}^n$, $y\in\mathbf{R}$. We consider the case

$$C_0(y)=\begin{cases} C_- & \text{for } y<0\\ C_h & \text{for } 0<y<h\\ C_+ & \text{for } y>h,\end{cases} \tag{1·2}$$

where $C_+,\ C_h,\ C_-$ and h are positive constants. On the other hand, the acoustic wave in a perturbed stratified fluid which we consider here is also governed by a similar wave equation

$$\frac{\partial^2 w}{\partial t^2}-C^2(z)\Delta w=0. \tag{1·3}$$

The sound speed $C(z)>0$ is assumed to satisfy the following conditions:

(C1) $C(z)$ is measurable;

(C2) $0<C_m<C(z)<C_M$ for some C_m and C_M;

(C3) $C(z)-C_0(y)=O(|z|^{-1-\delta})$, as $|z|\to\infty$, for some $0<\delta<1$.

The assumption (C3) means that the perturbation under consideration is of short-range class.

Write $\mu(z)=1/C^2(z)$ and $\mu_0(z)=1/C_0^2(y)$. We shall prove in the present paper that, if $C_h\geqslant\min(C_+,C_-)$ then there exists at most one function u in $H^2(\Omega(R))$ for all $R>0$ satisfying the equation

$$-\Delta u-\lambda\mu u=\mu f \tag{1·4}$$

in $\mathbf{R}^{n+1}$, and the Sommerfeld radiation condition of the form

$$\lim_{R\to\infty}\frac{1}{R}\int_{\Omega(R)}\left|\frac{\partial u}{\partial|z|}-i\sqrt{\lambda\mu_0}u\right|^2dz=0, \tag{1·5}$$

where $\lambda>0$ and f is a function in $L_2(\mathbf{R}^{n+1})$. Here, $\Omega(R)=\{z\in\mathbf{R}^{n+1}||z|<R\}$ and, for a domain V in $\mathbf{R}^{n+1}$ and a non-negative integer k, $H^k(V)$ denotes the usual Sobolev space with scalar product $(\cdot,\cdot)_{k,V}$ and norm $\|\cdot\|_{k,V}$. Note that $L_2(V)=H^0(V)$ and V,k will be omitted from the scalar product and norm when $V=\mathbf{R}^{n+1}$ and $k=0$.

For $\alpha\in\mathbf{R}$ and a non-negative integer k, we define the weighted Sobolev space $H^k_\alpha(V)$ by

$$H^k_\alpha(V):=\{v\in H^k(V\cap\Omega(R))\,\forall R>0|\,\langle z\rangle^\alpha\partial^\sigma v\in L_2(V),\ |\sigma|\leqslant k\}$$

with the weighted norm

$$\|v\|^2_{k,\alpha,V}=\sum_{|\sigma|\leqslant k}\int_V\langle z\rangle^{2\alpha}|\partial^\sigma v|^2dz,\quad \langle z\rangle=(1+|z|^2)^{\frac12},$$

where $\sigma=(\sigma_1,\ldots,\sigma_{n+1})$, $\sigma_j\geqslant 0$ is an integer, $\partial^\sigma=\partial_1^{\sigma_1}\cdots\partial_n^{\sigma_n}\partial_y^{\sigma_{n+1}}$, $\partial_j=\partial/\partial x_j$ and $\partial_y=\partial/\partial y$. We write $H^0_\alpha(V)=L_2^\alpha(V)$. We also define

$$B(V):=\{f\in L_2(V\cap\Omega(R))\ \forall R>0|\|f\|_{B(V)}<+\infty\}$$

with

$$\|f\|_{B(V)}=\sum_{m=1}^{\infty}\left[r_m\int_{V_m}|f|^2dz\right]^{\frac12}$$

and

$$B^*(V) := \{u \in L_2(V \cap \Omega(R))\ \forall R > 0 | \|u\|_{B^*(V)} < +\infty\}$$

with

$$\|u\|^2_{B^*(V)} = \sup_{R>1} R^{-1} \int_{V\cap\Omega(R)} |u|^2 dz,$$

where $V_m = \{z \in V | r_{m-1} < |z| < r_m\}$, $r_0 = 0$, $r_m = 2^{m-1}$, $m \geqslant 1$. (The spaces B, B^* were introduced by Agmon and Hörmander [**1**] for the whole space $\mathbf{R}^n$ to study boundary values of the resolvent of a constant, symmetric differential operator $P(D)$ where the symbol $P(\xi)$ is a polynomial having only simple zeros. The Fourier transform of $B(\mathbf{R}^n)$ is the Besov space $B_2^{\frac{1}{2},1}(\mathbf{R}^n)$ and that of $B^*(\mathbf{R}^n)$ is the Besov space $B_2^{-\frac{1}{2},\infty}(\mathbf{R}^n)$ see [**11**].) Note that for every $\alpha > \frac{1}{2}$ the continuous inclusions

$$L_2^\alpha(V) \subset B(V) \subset L_2^{\frac{1}{2}}(V) \subset L_2(V) \subset L_2^{-\frac{1}{2}}(V) \subset B^*(V) \subset L_2^{-\alpha}(V) \tag{1·6}$$

hold. For simplicity, we shall omit V from norms and integrals in the present paper when $V = \mathbf{R}^{n+1}$.

With the above notations, we now formulate the main results obtained in the present paper.

THEOREM 1·1. *Let* $u \in H^2(\Omega(R))$ *for all* $R > 0$ *satisfy the equation*

$$-\Delta u - \lambda\mu u = 0, \quad \lambda > 0 \tag{1·7}$$

and the radiation condition (1·5)*. Then* $u = 0$ *almost everywhere in* $\mathbf{R}^{n+1}$.

Remark 1·1. The radiation condition (1·5) characterizes the distorted (due to the stratification) spherical waves that propagate along all directions as in the non-stratified case, but rules out guided waves localized in the slab $0 < y < h$ (such waves propagate in the x-direction and their amplitude is exponentially decaying in the y-direction). Note that if $C_h < \min(C_+, C_-)$ the Pekeris profile (1·2) has an infinite number of guided waves [**23**, **24**]. However, in the case when $C_h \geqslant \min(C_+, C_-)$ there are no guided waves (see [**23**, **24**]) so that we have the following corollary of Theorem 1·1.

COROLLARY 1·1. *Let* $C_h \geqslant \min(C_+, C_-)$*. Then the problem* (1·4)–(1·5) *has at most one solution.*

The proof of Theorem 1·1 rests on the unique continuation principle [**14**, p. 65] and the following result due to Weder [**21**, lemma I].

LEMMA 1·1. *If* $u \in L_2(E(R))$ *satisfies* (1·7) *in distribution sense in* $E(R)$ *for some* $R > 0$*. Then* $u = 0$ *almost everywhere in* $E(R)$*, where* $E(R) = \{z \in \mathbf{R}^{n+1} | |z| > R\}$.

By the unique continuation principle and Lemma 1·1 it follows that in order to show Theorem 1·1 it is sufficient to prove that $u \in L_2(E(R))$ for some $R > 0$. This will be done using Mourre's commutator method [**4**, **15**, **16**].

Consider now the damped equation

$$-\Delta u_\epsilon - (\lambda + i\epsilon)\mu u_\epsilon = \mu f \quad \text{in } \mathbf{R}^{n+1}, \tag{1·8}$$

where $\epsilon \neq 0$ and $u_\epsilon \in H^2(\mathbf{R}^{n+1})$. For this equation the standard L_2-solution theory is applicable and from the Lax–Milgram theorem and elliptic regularity estimates we have the following result concerning the unique solvability of (1·8).

Lemma 1·2. *For any* $f \in L_2(\mathbf{R}^{n+1})$ *there exists a unique solution* u_ϵ *of* (1·8) *in the space* $H^2(\mathbf{R}^{n+1})$ *and* u_ϵ *satisfies the estimate*

$$\|u_\epsilon\|_{2,-\alpha} \leqslant C(\|f\|_{0,-\alpha} + \|u_\epsilon\|_{0,-\alpha}) \tag{1·9}$$

for any $\alpha \geqslant 0$, *where* C *is a positive constant independent of* ϵ, f *and* u_ϵ. (*For simplicity* C *will denote a constant which may have different meaning at different places.*)

For the solution u_ϵ we shall establish the following a priori $B - B^*$ estimate.

Theorem 1·2. *Let* $f \in B(\mathbf{R}^{n+1})$ *and* u_ϵ *be the solution of* (1·8). *Then*

$$\|\nabla u_\epsilon\|_{B^*} + \|u_\epsilon\|_{B^*} \leqslant C\|f\|_B. \tag{1·10}$$

Remark 1·2. Let $\mathscr{H}$ be the Hilbert space consisting of all functions in $L_2(\mathbf{R}^{n+1})$ with the scalar product

$$(w, v)_{\mathscr{H}} = \int w(z)\bar{v}(z)C^{-2}(z)dz.$$

As is easily seen, the perturbed acoustic propagator $H = -C^2(z)\Delta$ is a positive self-adjoint operator in $\mathscr{H}$ with domain

$$D(H) = \{v \in \mathscr{H} | \Delta v \in \mathscr{H}\} = H^2(\mathbf{R}^{n+1}).$$

Denote by $R(\xi; H)$, $\Im\xi \neq 0$, the resolvent of H, that is, $R(\xi; H) = (H - \xi)^{-1}$. Then the limiting absorption principle for H has been obtained previously in [**2**, **3**, **5**, **7**, **19**–**21**, **23**], which implies that

$$\|R(\lambda + i\epsilon; H)f\|_{-\alpha} \leqslant C\|f\|_\alpha, \tag{1·11}$$

for $\epsilon \neq 0$ and $\lambda > 0$, $R(\lambda + i\epsilon; H)f$ converges to $R(\lambda + i0; H)f$ strongly in $L_2^{-\alpha}(\mathbf{R}^{n+1})$ for $f \in L_2^\alpha(\mathbf{R}^{n+1})$ with $\alpha > \frac{1}{2}$ and $R(\lambda + i0; H)f$ is Hoelder continuous in λ. On the other hand, since the solution of (1·8) can be written as $u_\epsilon = R(\lambda + i\epsilon; H)f$, then by Theorem 1·2 we have

$$\|R(\lambda + i\epsilon; H)f\|_{B^*} \leqslant C\|f\|_B, \tag{1·12}$$

for $\epsilon \neq 0$, which implies the existence and the uniqueness of the weak-$*$ limit in $B^*(\mathbf{R}^{n+1})$ for $R(\lambda \pm i\epsilon; H)f$ as $\epsilon \downarrow 0$, when $f \in B(\mathbf{R}^{n+1})$, and $\lambda > 0$. This result follows from the above $B - B^*$ estimate, the density of $L_2^\alpha(\mathbf{R}^{n+1})$ in $B(\mathbf{R}^{n+1})$ for $\alpha > \frac{1}{2}$ and the existence of the boundary values $R(\lambda \pm i0; H)$ in the $L_2^\alpha - L_2^{-\alpha}$ topology for $\alpha > \frac{1}{2}$ (see above). This result is optimal in the sense that for each $\lambda > 0$, there is a dense open subset of $B(\mathbf{R}^{n+1})$ for which convergence to the weak limit can not be improved.

Theorem 1·2 gives the a priori $B - B^*$ estimate for u_ϵ, which plays an important role in proving the limiting absorption principle and in establishing existence of solutions to the problem (1·4)–(1·5). The proof of Theorem 1·2 is done by an operator theoretical approach based on Mourre's commutator method and is different from that proving the a priori estimates in [**2**, **7**, **19**–**21**, **23**]. A $B - B^*$ estimate was first obtained using Mourre's commutator method by Jensen and Perry [**11**] for a class of generalized N-body Schroedinger operators. Their argument is used in this paper to get a $B - B^*$ resolvent estimate for the unperturbed acoustic propagator.

Using this result, together with Theorem 1·1 and a compact argument, we are able to prove Theorem 1·2. We should remark that by using a commutator estimate for the perturbed acoustic propagator (see [**5**, **6**]), Theorem 1·2 follows directly from the argument in [**11**]. This, however, requires much stronger assumptions on the sound speed $C(z)$ than those made in this paper (see [**5**, **6**]).

The commutator method was first developed by Mourre [**15**] to prove the principle of limiting absorption for three-body Schroedinger operators and its application has been extended, for example, in [**8**, **10**, **13**, **16**, **18**] to various spectral problems of N-body Schroedinger operators and others. In these works, it has been used to prove the limiting absorption principle ([**16**]), to show the non-existence of positive eigenvalues ([**8**]), to study the resolvent smoothness as a function of energy ([**10**]), to establish the low frequency asymptotics of resolvents ([**13**, **18**]) and to prove the limiting amplitude principle ([**13**, **18**]). In the present work, it is seen that this remarkable method is also useful in determining the asymptotic property at infinity of the boundary values of resolvents. The asymptotic behavior at infinity is very important in the study of the associated boundary problems since from it a radiation condition can be obtained, which ensures the uniqueness of the solution to the problem. So the unique solvability of the boundary problem could be established.

The remaining sections are organized as follows. In Section 2 certain results are presented for the resolvent of the unperturbed acoustic propagator. Theorem 1·1 is proved in Section 3 and Section 4 is devoted to the proof of Theorem 1·2.

2. *Commutator method*

In this section we shall prepare fundamental results concerning the resolvent of the unperturbed acoustic propagator $\hat{H}_0 = -C_0^2(y)\Delta$. Similarly as in [**5**, **6**] we consider the unitarily equivalent operator of $\hat{H}_0$,

$$H_0 = U_0 \hat{H}_0 U_0^{-1},$$

where U_0 is an operator from $L_2(\mathbf{R}^{n+1}; C_0^{-2} dz)$ to $\mathscr{H}_0 = L_2(\mathbf{R}^{n+1}; dz) \equiv L_2(\mathbf{R}^{n+1})$ defined by

$$U_0 \psi = C_0^{-1} \psi, \quad \psi \in L_2(\mathbf{R}^{n+1}; C_0^{-2} dz).$$

Then $H_0 = -C_0 \Delta C_0$ is self-adjoint on $\mathscr{H}_0$ with domain

$$\mathscr{H}_2 \equiv D(H_0) = \{\psi \in \mathscr{H}_1 | C_0 \Delta C_0 \psi \in \mathscr{H}_0\},$$

where

$$\mathscr{H}_1 = \{\psi \in \mathscr{H}_0 | \nabla C_0 \psi \in \mathscr{H}_0\}.$$

It is noted that if $\psi \in \mathscr{H}_2$, then $C_0 \psi \in H^2(\mathbf{R}^{n+1}) \cap L_2(\mathbf{R}^{n+1}; C_0^{-2} dz)$ and $\psi \in \mathscr{H}_0$. Let $\eta \in C^\infty(\mathbf{R})$ be such that $0 \leqslant \eta \leqslant 1$, $\eta(t) = 1$ for $t < -h$ and $t > 2h$, $\eta(t) = 0$ for $-h/2 \leqslant t \leqslant 3h/2$, and let us define

$$A_0 = \tfrac{1}{2}(x \cdot \nabla_x + \nabla_x \cdot x + \eta(y) y \partial_y + \partial_y \eta(y) y). \tag{2·1}$$

Then A_0 is the generator of a dilation unitary group in $\mathbf{R}^{n+1}$. The following series of results have been established by [**5**, **6**].

THEOREM 2·1. *The pure point spectrum $\sigma_{pp}(H_0)$ of the operator H_0 is empty.*

Lemma 2·1. (*a*) *The commutator form* $[H_0, A_0] = H_0A_0 - A_0H_0$, *defined on* $D(A_0) \cap D(H_0)$, *extends to a bounded operator from* $H^2(\mathbf{R}^{n+1})$ *to* $\mathscr{H}_0$;

(*b*) *The commutator form* $[[H_0, A_0], A_0]$, *defined on* $D(A_0) \cap D(H_0)$, *extends to a bounded operator from* $H^2(\mathbf{R}^{n+1})$ *to* $H^{-2}(\mathbf{R}^{n+1})$, *where* $H^{-2}(\mathbf{R}^{n+1})$ *is the dual of* $H^2(\mathbf{R}^{n+1})$.

Lemma 2·2. *For any* $\lambda > 0$, *there exist a function* $g \in C_0^\infty(\mathbf{R}^+)$ *and a constant* $\beta > 0$ *such that* $g \equiv 1$ *on a small neighbourhood of* λ, $0 \leqslant g \leqslant 1$, *and*

$$M_g \equiv g(H_0)[H_0, A_0]g(H_0) \geqslant \beta g(H_0)^2, \tag{2·2}$$

where $\mathbf{R}^+$ *denotes the positive real axis.*

Lemma 2·3. *For any* $g \in C_0^\infty(\mathbf{R}^+)$, *the commutators* $[g(H_0), A_0]$ *and* $[M_g, A_0]$ *are bounded on* $\mathscr{H}_0$.

Lemma 2·4. *Let* $g \in C_0^\infty(\mathbf{R}^+)$. *Then for any non-negative* $\alpha \leqslant 1$, *the operator* $\langle z\rangle^{-\alpha} g(H_0)\langle A_0\rangle^\alpha$ *is bounded on* $\mathscr{H}_0$.

Remark 2·1. Theorem 2·1 was proved by [**6**, theorem 1·3] and [**21**, lemma I]. In fact, Theorem 2·1 is a special case of theorem 1·3 of [**6**] and lemma I of [**21**]. In [**6**, **21**], the absence of positive eigenvalues were proved for more general operators than H_0 here.

Remark 2·2. Lemmas 2·1–2·4 have been proved by [**5**, **6**]. It should be mentioned that these results were verified in [**5**, **6**] for more general operators H_0 and A_0 compared with ones considered here, and for our special operator H_0 the operator A_0 introduced in [**5**, **6**] can be taken as in (2·1).

Theorem 2·2. *Let* $G(\xi) = (H_0 - \xi)^{-1}$ *with* $\Im\xi \neq 0$.

(i) *If* $\alpha > \frac{1}{2}$, *then*

$$\|\langle z\rangle^{-\alpha} G(\lambda + i\tau)\langle z\rangle^{-\alpha}\| \leqslant C \tag{2·3}$$

for any $\lambda > 0$ *and* $\tau > 0$, *where* C *is a constant independent of* τ.

(ii) *For every* $\lambda > 0$ *and* $\alpha > \frac{1}{2}$, *the norm limit*

$$G(\lambda + i0) = \lim_{\tau \downarrow 0} G(\lambda + i\tau) \tag{2·4}$$

exists as a bounded operator from $L_2^\alpha(\mathbf{R}^{n+1})$ *to* $L_2^{-\alpha}(\mathbf{R}^{n+1})$.

This theorem was proved in [**5**, theorem 1·1] for $\lambda \in \mathbf{R}^+ \backslash \sigma_{pp}(\hat{H}_0^0)$, where $\hat{H}_0^0 = -C_0^2(y)\partial^2/\partial y^2$ and $\sigma_{pp}(\hat{H}_0^0)$ is the pure point spectrum of the operator H_0^0. From [**6**, theorem 2·1(a)] it follows that $\sigma_{pp}(\hat{H}_0^0) = \varnothing$ and hence Theorem 2·2 is true. It should be mentioned that in [**5**, **6**] more general operators H_0 were considered.

The following result can be easily proved by following exactly the same argument as used by Jensen and Perry in [**11**].

Theorem 2·3. *Let* $\lambda > 0$. *Then*

$$\sup_{\tau \neq 0} \|G(\lambda + i\tau)f\|_{B^*} \leqslant C(\lambda)\|f\|_B \tag{2·5}$$

for any $f \in B(\mathbf{R}^{n+1})$, *where* $C(\lambda)$ *is independent of* f *and can be chosen uniform in* λ *running over a fixed compact subset of* $\mathbf{R}^+$.

For any $\lambda > 0$, take $g \in C_0^\infty(\mathbf{R}^+)$ such that Lemma 2·2 holds and set $M = M_g$. By Lemma 2·2, M is non-negative definite and hence there exists a inverse

$$G(\epsilon, \xi) = (H_0 - \xi - i\epsilon M)^{-1}$$

of $H_0 - \xi - i\epsilon M$ for $\epsilon \geqslant 0$ and $\xi = \lambda + i\tau$ with $\tau > 0$. By Lemmas 2·1–2·4, the following lemma can be proved in exactly the same way as in the proof of lemmas 7·3, 7·6, 7·7 and theorem 7·8 in [**16**] (see also [**17**, propositions 2·5–2·7]).

LEMMA 2·5. *For sufficiently small* ϵ_0, $G(\epsilon, \xi)$ *is bounded on* $\mathscr{H}_0$ *for* $0 < \epsilon \leqslant \epsilon_0$ *and* $0 < \tau \leqslant 1$; *and*

$$\text{(i)} \qquad \|G(\epsilon,\xi)\langle z\rangle^{-\alpha}\|, \quad \|G^*(\epsilon,\xi)\langle z\rangle^{-\alpha}\| \leqslant C\epsilon^{-\frac{1}{2}}; \tag{2·6}$$

$$\text{(ii)} \qquad \|\langle A_0\rangle^{-\alpha} G(\epsilon,\xi)\|, \quad \|\langle A_0\rangle^{-\alpha} G^*(\epsilon,\xi)\| \leqslant C\epsilon^{-\frac{1}{2}}; \tag{2·7}$$

$$\text{(iii)} \qquad \|G(\epsilon,\xi)\|, \quad \|G^*(\epsilon,\xi)\| \leqslant C\epsilon^{-1}; \tag{2·8}$$

$$\text{(iv)} \qquad \|(1-g(H_0))G(\epsilon,\xi)\| \leqslant C, \quad \|(1-g(H_0))G^*(\epsilon,\xi)\| \leqslant C; \tag{2·9}$$

where $\alpha > \frac{1}{2}$, C *is a constant independent of* ϵ *and* τ, G^* *denotes the conjugate of* G *and* $\|\cdot\|$ *denotes the operator norm when considered as an operator from* $\mathscr{H}_0$ *into itself.*

Remark 2·3. The estimates (2·8) and (2·9) remain true if the operator norm $\|\cdot\|$ is replaced by the norm as operators from $\mathscr{H}_0$ into $H^2(\mathbf{R}^{n+1})$ (see lemma 7·3(d) in [**16**]).

Remark 2·4. The norm limit

$$G(\lambda + i0) = \lim_{\epsilon \downarrow 0} G(\epsilon, \lambda) \tag{2·10}$$

holds as a bounded operator from $L_2^\alpha(\mathbf{R}^{n+1})$ to $L_2^{-\alpha}(\mathbf{R}^{n+1})$ and the estimates (2·6)–(2·9) remain true for $0 < \epsilon \leqslant \epsilon_0$ and $\xi = \lambda$. These results can be easily verified in exactly the same way as in [**17**, proposition 2·10].

PROPOSITION 2·1. *If* $\phi \in L_2^\alpha(\mathbf{R}^{n+1})$ *and* $\Im(\phi, G(\lambda + i0)\phi) = 0$, *then* $G(\lambda + i0)\phi \in L_2^{-\beta}(\mathbf{R}^{n+1})$ *for any* $\beta > 0$, *where* $\alpha > \frac{1}{2}$.

The idea of proof is similar to that used by Tamura [**17**, proposition 2·11] and Iwashita [**9**, proposition 2·7], where the case when $\alpha = 1$ has been proved, and the proof will be done through several lemmas.

LEMMA 2·6. *For any* $\psi \in \mathscr{H}_0$ *and* $\alpha > \frac{1}{2}$ *we have*

$$\left|\frac{d}{d\epsilon}(\psi, X_\alpha(\epsilon)G(\epsilon,\lambda)X_\alpha(\epsilon)\psi)\right| \leqslant C_\psi \epsilon^{\alpha-1}(\|G(\epsilon,\lambda)X_\alpha(\epsilon)\psi\| + \|G^*(\epsilon,\lambda)X_\alpha(\epsilon)\psi\| + C_\psi), \tag{2·11}$$

where $X_\alpha(\epsilon) = \langle z\rangle^{-\alpha}\langle \epsilon z\rangle^{\alpha-1}$ *and* $C_\psi = C\|\psi\|$ *with a positive constant* C.

Proof. The argument is similar to that used in the proof of lemma 2·8 in [**8**]. For brevity, we write $G(\epsilon)$ for $G(\epsilon, \lambda)$. Let $F(\epsilon) = X_\alpha(\epsilon)G(\epsilon)X_\alpha(\epsilon)$. We have $(d/d\epsilon)F(\epsilon) = P_1 + P_2 + Q$, where

$$P_1 = [(d/d\epsilon)X_\alpha(\epsilon)]G(\epsilon)X_\alpha(\epsilon) = (\alpha - 1)\epsilon|z|^2\langle\epsilon z\rangle^{-2}F(\epsilon),$$
$$P_2 = X_\alpha(\epsilon)G(\epsilon)[(d/d\epsilon)X_\alpha(\epsilon)] = (\alpha - 1)\epsilon F(\epsilon)|z|^2\langle\epsilon z\rangle^{-2},$$
$$Q = iX_\alpha(\epsilon)G(\epsilon)MG(\epsilon)X_\alpha(\epsilon).$$

We decompose Q as

$$Q = Q_1 + Q_2 + Q_3 + Q_4,$$

where

$$\begin{aligned} Q_1 &= -iX_\alpha(\epsilon)G(\epsilon)(1 - g(H_0))[H_0, A_0](1 - g(H_0))G(\epsilon)X_\alpha(\epsilon), \\ Q_2 &= -iX_\alpha(\epsilon)G(\epsilon)(1 - g(H_0))[H_0, A_0]g(H_0)G(\epsilon)X_\alpha(\epsilon) \\ &\quad -iX_\alpha(\epsilon)G(\epsilon)g(H_0)[H_0, A_0](1 - g(H_0))G(\epsilon)X_\alpha(\epsilon), \\ Q_3 &= iX_\alpha(\epsilon)G(\epsilon)[H_0 - i\epsilon M - \lambda, A_0]G(\epsilon)X_\alpha(\epsilon), \\ Q_4 &= -\epsilon X_\alpha(\epsilon)G(\epsilon)[M, A_0]G(\epsilon)X_\alpha(\epsilon). \end{aligned}$$

Thus, we have

$$(d/d\epsilon)(\psi, F(\epsilon)\psi) = (\psi, P_1\psi) + (\psi, P_2\psi) + \sum_{j=1}^{4}(\psi, Q_j\psi). \tag{2·12}$$

A direct calculation yields

$$|(\psi, P_1\psi)| \leqslant C_\psi \epsilon^{\alpha-1}\|G(\epsilon)X_\alpha(\epsilon)\psi\|, \tag{2·13}$$

$$|(\psi, P_2\psi)| \leqslant C_\psi \epsilon^{\alpha-1}\|G^*(\epsilon)X_\alpha(\epsilon)\psi\|. \tag{2·14}$$

Since

$$(1 - g(H_0))G(\epsilon) = (1 - g(H_0))G(\lambda)\{i\epsilon MG(\epsilon) + 1\}, \tag{2·15}$$

then from (2·8), (2·9), (2·15), Lemma 2·1(a) and Remarks 2·3 and 2·4 it follows that

$$|(\psi, Q_1\psi)| \leqslant C\|\psi\|\|G^*(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C_\psi\|G^*(\epsilon)X_\alpha(\epsilon)\psi\|, \tag{2·16}$$

$$|(\psi, Q_2\psi)| \leqslant C_\psi\|(H_0 + i)g(H_0)G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C_\psi\|G(\epsilon)X_\alpha(\epsilon)\psi\|, \tag{2·17}$$

where use has been made of the commutability of $1 - g(H_0)$ with $G(\epsilon)$. Expanding the commutator in Q_3 and taking into account the definition of $G(\epsilon)$, we find

$$\begin{aligned} (\psi, Q_3\psi) &= (\psi, [X_\alpha(\epsilon)A_0G(\epsilon)X_\alpha(\epsilon) - X_\alpha(\epsilon)G(\epsilon)A_0X_\alpha(\epsilon)]\psi) \\ &= (\psi, X_\alpha(\epsilon)A_0G(\epsilon)X_\alpha(\epsilon)\psi) - (X_\alpha(\epsilon)A_0G^*(\epsilon)X_\alpha(\epsilon)\psi, \psi) \\ &\equiv I_1 + I_2. \end{aligned} \tag{2·18}$$

Estimation of I_1 is as follows:

$$\begin{aligned} |I_1| \leqslant \|\psi\|\|X_\alpha(\epsilon)A_0(H_0 + i)^{-1}\|[\|(H_0 + i)g(H_0)G(\epsilon)X_\alpha(\epsilon)\psi\| \\ +\|(H_0 + i)(1 - g(H_0))G(\epsilon)X_\alpha(\epsilon)\psi\|]. \end{aligned}$$

We have

$$\|X_\alpha(\epsilon)A_0(H_0 + i)^{-1}\| \leqslant C\epsilon^{\alpha-1},$$

$$\|(H_0 + i)g(H_0)G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant \|G(\epsilon)X_\alpha(\epsilon)\psi\|.$$

On the other hand, by (2·15) and the fact that $\|(H_0 + i)(1 - g(H_0))G(\lambda)\| \leqslant C$, we obtain

$$\|(H_0 + i)(1 - g(H_0))G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C(\epsilon\|G(\epsilon)X_\alpha(\epsilon)\psi\| + C_\psi).$$

Hence

$$|I_1| \leqslant C_\psi\epsilon^{\alpha-1}(C_\psi + \|G(\epsilon)X_\alpha(\epsilon)\psi\|). \tag{2·19}$$

Similarly,

$$|I_2| \leqslant C_\psi \epsilon^{\alpha-1}(C_\psi + \|G^*(\epsilon)X_\alpha(\epsilon)\psi\|). \tag{2·20}$$

We thus deduce from (2·18), (2·19) and (2·20) that

$$|(\psi, Q_3\psi)| \leqslant C_\psi \epsilon^{\alpha-1}(C_\psi + \|G(\epsilon)X_\alpha(\epsilon)\psi\| + \|G^*(\epsilon)X_\alpha(\epsilon)\psi\|). \tag{2·21}$$

Finally we evaluate by using Lemma 2·3 and (2·8),

$$|(\psi, Q_4\psi)| \leqslant C_\psi \|G(\epsilon)X_\alpha(\epsilon)\psi\|. \tag{2·22}$$

Combining (2·13), (2·14), (2·16), (2·17), (2·21) and (2·22) leads to the estimate (2·11). The proof is complete.

Lemma 2·7. *Let* $\psi \in \mathscr{H}_0$ *and* $\Im(\psi, X_\alpha(0)G(\lambda + i0)X_\alpha(0)\psi) = 0$. *Then*

$$\|G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C\epsilon^{-\eta}$$

for any $\eta > 0$, *where* $\alpha > \frac{1}{2}$.

This lemma can be proved using Lemmas 2·5 and 2·6 in exactly the same way as in the proof of lemma 2·9 in [**9**].

Lemma 2·8. *Let* $\beta > 0$. *Then*

$$\|G(\epsilon,\xi)X_\beta(\epsilon)\|, \quad \|G^*(\epsilon,\xi)X_\beta(\epsilon)\|, \quad \|X_\beta(\epsilon)G(\epsilon,\xi)\|, \quad \|X_\beta(\epsilon)G^*(\epsilon,\xi)\| \leqslant C\epsilon^{\beta-1-\eta} \tag{2·23}$$

for any $\eta > 0$, *where* $\xi = \lambda + i\tau$ *with* $\lambda > 0$ *and* $0 < \tau \leqslant 1$, *and* C *is a constant independent of* ϵ, ξ, β *and* η.

Proof. We prove the first estimate in (2·23) and others can be proved similarly. For brevity, we write $G(\epsilon)$ for $G(\epsilon,\xi)$. Let $F(\epsilon) = X_\beta(\epsilon)G(\epsilon)X_\beta(\epsilon)$.

First of all, using exactly the same argument as in the proof of lemma 7·6 in [**16**] and Lemma 2·5, it can be easily shown that

$$\|G(\epsilon)X_\beta(\epsilon)\|, \quad \|G^*(\epsilon)X_\beta(\epsilon)\| \leqslant C(1 + \epsilon^{-\frac{1}{2}}\|F(\epsilon)\|^{\frac{1}{2}}). \tag{2·24}$$

Next, from the proof of Lemma 2·6 it can be seen that the inequality (2·11) also holds for λ and α replaced by ξ and β, respectively. This, together with (2·24), implies

$$\left\|\frac{d}{d\epsilon}F(\epsilon)\right\| \leqslant C\epsilon^{\beta-1}(1 + \epsilon^{-\frac{1}{2}}\|F(\epsilon)\|^{\frac{1}{2}}). \tag{2·25}$$

Since, by (2·8), $\|F(\epsilon)\| \leqslant \|G(\epsilon)\| \leqslant C\epsilon^{-1}$, then it follows from (2·25) that $\|(d/d\epsilon)F(\epsilon)\| \leqslant C\epsilon^{\beta-2}$. Integrating the above inequality from ϵ to ϵ_0 leads to the result that $\|F(\epsilon)\| \leqslant C\epsilon^{\beta-1}$. From this and repeating the above argument n times we arrive at

$$\|F(\epsilon)\| \leqslant C\epsilon^{-1+\beta+\beta/2+\cdots+\beta/2^n} = C\epsilon^{2\beta-1-\beta/2^n}.$$

This, together with (2·24), implies that

$$\|G(\epsilon)X_\beta(\epsilon)\|, \quad \|G^*(\epsilon)X_\beta(\epsilon)\| \leqslant C\epsilon^{\beta-1-\beta/2^{n+1}}.$$

For any $\eta > 0$, choose n such that $\eta > \beta/2^{n+1}$ to obtain

$$\|G(\epsilon)X_\beta(\epsilon)\|, \quad \|G^*(\epsilon)X_\beta(\epsilon)\| \leqslant C\epsilon^{\beta-1-\eta}.$$

The lemma is proved.

Proof of Proposition 2·1. Let $\phi = X_\alpha(0)\psi$ with $\psi \in \mathscr{H}_0$ and let $\beta \ll 1$. Put $F(\epsilon) = X_\beta(\epsilon)G(\epsilon)X_\alpha(\epsilon)$, where $G(\epsilon) = G(\epsilon, \lambda)$. We then prove

$$\|F(\epsilon)\| \leqslant C, \tag{2·26}$$

for a constant C independent of ϵ, β and α. If (2·26) is true, then taking $\epsilon = 0$ in (2·26) implies that $G(\lambda + i0)\phi \in L_2^{-\beta}(\mathbf{R}^{n+1})$, which proves the proposition.

To prove (2·26), we differentiate $F(\epsilon)$ with respect to ϵ and obtain $(d/d\epsilon)F(\epsilon) = P_1 + P_2 + Q$, with

$$\begin{aligned} P_1 &= (\beta - 1)h(z)X_\beta(\epsilon)G(\epsilon)X_\alpha(\epsilon)\psi, \\ P_2 &= (\alpha - 1)X_\beta(\epsilon)G(\epsilon)h(z)X_\alpha(\epsilon)\psi, \\ Q &= iX_\beta(\epsilon)G(\epsilon)MG(\epsilon)X_\alpha(\epsilon)\psi, \end{aligned}$$

where $h(z) = \epsilon|z|^2\langle\epsilon z\rangle^{-2}$. By Lemma 2·7, we find

$$\|P_1\| \leqslant C\epsilon^{\beta-1}\|G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C\epsilon^{\beta-1-\eta} \tag{2·27}$$

for any $\eta > 0$. It is seen from Lemmas 2·7 and 2·8 that

$$\|Q\| \leqslant C\|X_\beta(\epsilon)G(\epsilon)\|\|G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C\epsilon^{\beta-1-\eta}, \tag{2·28}$$

since M is bounded on $\mathscr{H}_0$. To estimate P_2, we write P_2 as follows:

$$\begin{aligned} P_2 = (\alpha - 1)\{&X_\beta(\epsilon)h(z)G(\epsilon)X_\alpha(\epsilon)\psi + iX_\beta(\epsilon)G(\epsilon)(\epsilon[M, h])G(\epsilon)X_\alpha(\epsilon)\psi \\ &+X_\beta(\epsilon)G(\epsilon)[h, H_0]G(\epsilon)X_\alpha(\epsilon)\psi\} \equiv (\alpha - 1)\{I_1 + I_2 + I_3\}. \end{aligned} \tag{2·29}$$

Since M is bounded on $\mathscr{H}_0$, then $\|\epsilon[M, h]\| \leqslant C$ and therefore, by Lemmas 2·7 and 2·8, we have

$$I_1 \leqslant C\|X_\beta(\epsilon)h(z)\|\|G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C\epsilon^{\beta-1-\eta}, \tag{2·30}$$

$$I_2 \leqslant C\|X_\beta(\epsilon)G(\epsilon)\|\|G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C\epsilon^{\beta-1-\eta}. \tag{2·31}$$

A direct calculation of the commutator $[h, H_0]$ gives

$$[h, H_0] = C_0^2(\Delta h + 2\nabla h \cdot \nabla) = C_0^2\{\Delta h + 2h_1(z)\tilde{A}_0 - (n + 1)h_1(z)\},$$

where $|\Delta h| \leqslant C\epsilon$, $h_1(z) = \epsilon\langle\epsilon z\rangle^{-2}[1 - \epsilon h(z)]$ with $|h_1(z)| \leqslant C\epsilon$ and $|\langle z\rangle h_1(z)| \leqslant C$, and $\tilde{A}_0 = z \cdot \nabla + (n + 1)/2$. Noting that $\tilde{A}_0 = A_0 - (\eta(y) - 1)y\partial_y - [(\eta(y) - 1) + y\eta'(y)]/2$, we have

$$\begin{aligned} [h, H_0] = C_0^2\{&\Delta h - h_1(z)[(\eta(y) - 1) + y\eta'(y) + n + 1] \\ &-2h_1(z)(\eta(y) - 1)y\partial_y + 2h_1(z)A_0\} \equiv C_0^2\{J_1 + J_2 + J_3 + J_4\}. \end{aligned}$$

Thus, I_3 can be decomposed as $I_3 = \sum_{j=1}^4 I_{3j}$, where

$$I_{3j} = X_\beta(\epsilon)G(\epsilon)C_0^2J_jG(\epsilon)X_\alpha(\epsilon)\psi, \qquad (j = 1, 2, 3, 4).$$

It follows from Lemmas 2·7 and 2·8 that

$$\|I_{3j}\| \leqslant C\epsilon\|X_\beta(\epsilon)G(\epsilon)\|\|G(\epsilon)X_\alpha(\epsilon)\psi\| \leqslant C\epsilon^{\beta-\eta}, \qquad (j = 1, 2). \tag{2·32}$$

Using Lemmas 2·5(iii) and 2·8 in conjunction with Remark 2·3 it is found that

$$\|I_{33}\| \leqslant C\epsilon\|X_\beta(\epsilon)G(\epsilon)\|\|G(\epsilon)\|_{0\to 2} \leqslant C\epsilon^{\beta-1-\eta}, \tag{2·33}$$

where $\|\cdot\|_{0\to 2}$ denotes the operator norm as operators from $\mathscr{H}_0$ into $H^2(\mathbf{R}^{n+1})$. To estimate I_{34}, we make use of Lemma 2·8 and get

$$\begin{aligned}\|I_{34}\| &\leqslant C\|X_\beta(\epsilon)G(\epsilon)\|\|h_1(z)A_0G(\epsilon)X_\alpha(\epsilon)\psi\| \\ &\leqslant C\epsilon^{\beta-1-\eta}\|\langle z\rangle^{-1}A_0G(\epsilon)X_\alpha(\epsilon)\psi\| \equiv C\epsilon^{\beta-1-\eta}J. \quad (2{\cdot}34)\end{aligned}$$

On the other hand, using Lemma 2·4, (2·15) and Lemma 2·7, we obtain

$$\begin{aligned}J &\leqslant \|\langle z\rangle^{-1}A_0g(H_0)G(\epsilon)X_\alpha(\epsilon)\psi\| + \|\langle z\rangle^{-1}A_0(1-g(H_0))G(\epsilon)X_\alpha(\epsilon)\psi\| \\ &\leqslant C\{\|G(\epsilon)X_\alpha(\epsilon)\psi\| + \|(H_0+i)(1-g(H_0))G(\epsilon)X_\alpha(\epsilon)\psi\|\} \\ &\leqslant C\{\|G(\epsilon)X_\alpha(\epsilon)\psi\| + 1\} \leqslant C\epsilon^{-\eta}. \quad (2{\cdot}35)\end{aligned}$$

Now combining the estimates (2·32), (2·33), (2·34) and (2·35) it is obtained that $\|I_3\| \leqslant C\epsilon^{\beta-1-2\eta}$. This inequality together with (2·30) and (2·31) gives the result that $\|P_2\| \leqslant C\epsilon^{\beta-1-2\eta}$. Hence, it is seen on combining the above inequality with (2·27) and (2·28) that $\|(d/d\epsilon)F(\epsilon)\| \leqslant C\epsilon^{\beta-1-2\eta}$. Take η such that $\beta-3\eta>0$ for arbitrarily fixed β to obtain that $\|(d/d\epsilon)F(\epsilon)\| \leqslant C\epsilon^{\eta-1}$. Integrating the above inequality with respect to ϵ from ϵ to ϵ_0 yields

$$\|F(\epsilon)\| \leqslant \|F(\epsilon_0)\| + 1 \leqslant \|G(\epsilon_0)\| + 1,$$

which, together with (2·8), implies (2·26). The proposition is thus proved.

3. *Proof of Theorem* 1·1

In this section we prove Theorem 1·1. In view of the unique continuation principle and Lemma 1·1 it is enough to show that $u \in \mathscr{H}_0$. This will be done through several lemmas.

LEMMA 3·1. *We have* $(|u| + |\nabla u|) \in B^*(\mathbf{R}^{n+1})$ *and*

$$\lim_{R\to\infty}\frac{1}{R}\int_{\Omega(R)}(|\partial_r u|^2 + |u|^2)dx = 0, \quad (3{\cdot}1)$$

where $\partial_r u = \hat{z}\cdot\nabla u$ *and* $\hat{z} = z/|z|$.

This lemma can be easily proved by multiplying (1·7) by $\bar{u}$, integrating over $\Omega(R)$, taking the imaginary part of the both sides of the equation thus obtained and making use of (1·5) and Schwarz's inequality.

LEMMA 3·2. $u \in L_2^{-\beta}(\mathbf{R}^{n+1})$ *for any* $\beta > 0$.

Proof. Take $\phi \in C^\infty(\mathbf{R}^+)$ so that $0 \leqslant \phi \leqslant 1$, $\phi(t) = 1$ for $0 \leqslant t \leqslant 1$ and $\phi(t) = 0$ for $t \geqslant 2$. For $R > 1$, set $\phi_R(z) = \phi(|z|/R)$ and $u_R(z) = \phi_R(z)u(z)$. Then $u_R \in H^2(\mathbf{R}^{n+1})$ and from (1·7) we have

$$-\Delta u_R - \lambda\mu_0 u_R = g_R \quad \text{in } \mathbf{R}^{n+1}, \quad (3{\cdot}2)$$

where $g_R = \phi_R g - 2\partial_r\phi_R\partial_r u - u\Delta\phi_R$ and $g = \lambda(\mu-\mu_0)u$. By Lemma 3·1 and (C3), $g \in L_2^\alpha(\mathbf{R}^{n+1})$ with $\alpha = \frac{1}{2} + \delta/2$. Since u_R has compact support in $\mathbf{R}^{n+1}$, then $v_R = C_0^{-1}u_R \in D(H_0) \subset \mathscr{H}_0$ and (3·2) is equivalent to

$$(H_0 - \lambda)v_R = f_R, \quad (3{\cdot}3)$$

where $f_R = C_0 g_R$. It follows from (3·3) that $v_R = G(\lambda + i\epsilon)[f_R - i\epsilon v_R]$, which, along with Theorem 2·2(ii) and the fact that f_R and v_R have compact support, gives

$$v_R = \lim_{\epsilon \downarrow 0} G(\lambda + i\epsilon)[f_R - i\epsilon v_R] = G(\lambda + i0) f_R \tag{3·4}$$

strongly in $L_2^{-\alpha}(\mathbf{R}^{n+1})$.

Set $v = G(\lambda + i0)f$ with $f = C_0 g$. Then by (1·6) and Theorems 2·2(ii) and 2·3 we find

$$\|v_R - v\|_{-\alpha} \leqslant \|v_R - v\|_{B^*} \leqslant C\|f_R - f\|_B \tag{3·5}$$

for some constant $C > 0$. Since by the definition of ϕ_R and $B(\mathbf{R}^{n+1})$

$$\begin{aligned}\|f_R - f\|_B^2 &\leqslant C\left[\|(\phi_R - 1)g\|_B^2 + \int_{\Omega(R,2R)} \frac{r}{R^2}(|\partial_r u|^2 + |u|^2)dx\right] \\ &\leqslant C\left[\|(\phi_R - 1)g\|_B^2 + \frac{1}{2R}\int_{\Omega(2R)} (|\partial_r u|^2 + |u|^2)dx\right],\end{aligned} \tag{3·6}$$

where $\Omega(R_1, R_2) = \{z \in \mathbf{R}^{n+1} | R_1 < |z| < R_2\}$. The first term on the right-hand side of the last inequality of (3·6) tends to zero as $R \to \infty$ and the second one goes to zero as well by use of (3·1). This in conjunction with (3·5) and (3·6) implies that v_R tends to v strongly in $L_2^{-\alpha}(\mathbf{R}^{n+1})$. However, by the definition of v_R it is seen that v_R converges to $C_0^{-1}u$ strongly in $L_2^{-\alpha}(\mathbf{R}^{n+1})$. Hence, $u = C_0 v = C_0 G(\lambda + i0)f$. On the other hand, since u satisfies the equation

$$-\Delta u - \lambda\mu_0 u = g \quad \text{in } \mathbf{R}^{n+1},$$

it then follows by using (3·1) that

$$\Im(f, G(\lambda + i0)f) = \Im(g, u) = -\liminf_{R\to\infty} \Im \int_{S(R)} \bar{u}\partial_r u ds = 0.$$

This together with Proposition 2·1 implies

$$u = C_0 G(\lambda + i0) f \in L_2^{-\beta}(\mathbf{R}^{n+1})$$

for any $\beta > 0$, since $f = C_0 g \in L_2^{\alpha}(\mathbf{R}^{n+1})$ and $\alpha > \frac{1}{2}$. This proves Lemma 3·2.

Remark 3·1. By Lemma 3·2 and elliptic regularity estimates it can be shown that $u \in H^2_{-\beta}(\mathbf{R}^{n+1})$ for any $\beta > 0$.

Lemma 3·3. *$u \in \mathscr{H}_0$.*

Proof. Let $\phi_\tau(z;\alpha) = (1 + \tau|z|^2)^{-\alpha/2}$ with $0 < \tau, \alpha \leqslant 1$ and $v_\tau(z) = \phi_\tau(z;\alpha)u(z)$. Then $v_\tau \in H^2(\mathbf{R}^{n+1})$, by Remark 3·1, and satisfies

$$-\Delta v_\tau - \lambda\mu_0 v_\tau = h_1, \tag{3·7}$$

where

$$h_1(z) = \phi_\tau g + \frac{2\alpha\tau|z|}{1 + \tau|z|^2}\partial_r v_\tau + \left\{\frac{3\alpha\tau}{1 + \tau|z|^2} - \frac{\alpha\tau^2(2 - \alpha)|z|^2}{(1 + \tau|z|^2)^2}\right\} v_\tau.$$

Set $v = C_0^{-1} v_\tau$. Then

$$H_0 v - \lambda v = h, \tag{3·8}$$

where $h = C_0 h_1$.

Take $g_0 \in C_0^\infty(\mathbf{R}^+)$ such that Lemma 2·2 holds. Set $f_0 = 1 - g_0$. Since the operator $G(\lambda + i0)f_0(H_0)$ is bounded, it follows from (3·8) that

$$\|f_0(H_0)v\| \leqslant C\|h\|. \tag{3·9}$$

Next, since $g_0(H_0)v$ solves the equation

$$(H_0 - \lambda)g_0(H_0)v = g_0(H_0)h, \tag{3·10}$$

we obtain by using Lemma 2·2

$$\beta\|g_0(H_0)v\|^2 \leqslant ([H_0, A_0]w, w), \tag{3·11}$$

where $w = g_0(H_0)v$. Take $\psi \in C_0^\infty(\mathbf{R}^+)$ so that $0 \leqslant \psi \leqslant 1$, $\psi(t) = 1$ for $0 \leqslant t \leqslant 1$ and $\psi(t) = 0$ for $t \geqslant 2$. For $R > 1$, set $\psi_R(z) = \psi(|z|/R)$. Integrating by parts and utilizing the definitions of $[H_0, A_0]$ and A_0 yields that

$$([H_0, A_0]w, w) = \lim_{R\to\infty}([H_0, A_0]w, \psi_R w) = 2\lim_{R\to\infty}\Re(H_0 w, \psi_R A_0 w), \tag{3·12}$$

$$\lim_{R\to\infty}\Re(w, \psi_R A_0 w) = 0. \tag{3·13}$$

Using (3·10)–(3·13) it is derived that

$$\begin{aligned}\beta\|g_0(H_0)v\|^2 \leqslant 2\lim_{R\to\infty}\Re(H_0 w, \psi_R A_0 w) &= 2\lim_{R\to\infty}\Re(g_0(H_0)h, \psi_R A_0 w)\\ &= 2\lim_{R\to\infty}\Re(h, g_0(H_0)A_0(\psi_R w)).\end{aligned} \tag{3·14}$$

Lemma 2·4 implies

$$\begin{aligned}2|\Re(h, g_0(H_0)A_0(\psi_R w))| &\leqslant 2\|\langle z\rangle h\|\|\langle z\rangle^{-1}g_0(H_0)A_0\|\|\psi_R w\|\\ &\leqslant \sigma\|\psi_R w\|^2 + C\|\langle z\rangle h\|^2\end{aligned} \tag{3·15}$$

for any $\sigma > 0$. Take $\sigma = \beta/2$ and make use of (3·14) and (3·15) to obtain

$$\|w\| \leqslant C\|\langle z\rangle h\|. \tag{3·16}$$

Now combining (3·9) and (3·16) we deduce

$$\|v\| \leqslant \|g_0(H_0)v\| + \|w\| \leqslant C\|\langle z\rangle h\| \leqslant C\{\|\langle z\rangle\phi_\tau g\| + \alpha(\|\partial_r v_\tau\| + \|v_\tau\|)\}. \tag{3·17}$$

On taking into account (C3) and the definition of g it is seen that for large $R > 0$

$$\|\langle z\rangle\phi_\tau g\| \leqslant CR^{-\delta}\|v_\tau\| + C_R\|v_\tau\|_{0,\Omega(R)} \tag{3·18}$$

with C independent of R. Utilizing (C2) and (3·7) we obtain

$$\|\nabla v_\tau\| \leqslant C(\|h_1\| + \|v_\tau\|) \leqslant C(T^{-1}\|\partial_r v_\tau\| + C_T\|\partial_r v_\tau\|_{0,\Omega(T)} + \|v_\tau\|)$$

for large $T > 0$. The first term on the right-hand side of the above inequality can be incorporated into the left-hand side for sufficiently large T and we have

$$\|\nabla v_\tau\| \leqslant C(\|v_\tau\| + \|\partial_r v_\tau\|_{0,\Omega(T)}). \tag{3·19}$$

As a result of (3·17), (3·18) and (3·19), we find

$$\|v\| \leqslant C(\alpha + R^{-\delta})\|v\| + C_R\|v\|_{1,\Omega(R)}$$

for $R \geqslant T$, where use has been made of (C2) and the fact that $v = C_0^{-1}v_\tau$. Taking R

large enough and α small enough yields $\|v\| \leqslant C_R\|v\|_{1,\Omega(R)}$. Letting $\tau \to 0$ we arrive at the result

$$\|u\| \leqslant C_R\|u\|_{1,\Omega(R)} < \infty,$$

that is, $u \in \mathscr{H}_0$. The proof is complete.

Proof of Theorem 1·1. Lemmas 3·3 and 1·1 combined with the unique continuation principle imply that $u = 0$ almost everywhere in $\mathbf{R}^{n+1}$, which proves the theorem.

4. *A priori* $B - B^*$ *estimates*

In this section we make use of Theorem 1·1 to prove Theorem 1·2. To this end, let $u_\epsilon \in H^2(\mathbf{R}^{n+1})$, $0 < \epsilon \leqslant 1$, be the solution of (1·8). We then prove the following result first.

Theorem 4·1. *Let* $f \in B(\mathbf{R}^{n+1})$. *Then*

$$\|\nabla u_\epsilon\|_{B^*} + \|u_\epsilon\|_{B^*} \leqslant C\{\|f\|_B + \|u_\epsilon\|_{0,\Omega(T)}\} \tag{4·1}$$

for some fixed $T > 0$.

Proof. Let $f_\epsilon = (\lambda + i\epsilon)(\mu - \mu_0)u_\epsilon$ and set $v_\epsilon = C_0^{-1}u_\epsilon$. Then $v_\epsilon \in D(H_0)$ and

$$H_0 v_\epsilon - (\lambda + i\epsilon)v_\epsilon = \hat{f}, \tag{4·2}$$

where $\hat{f} = C_0(\mu f + f_\epsilon)$. We thus have

$$v_\epsilon = G(\lambda + i\epsilon)\hat{f}.$$

By Theorem 2·3, it follows that

$$\|v_\epsilon\|_{B^*} = \|G(\lambda + i\epsilon)\hat{f}\|_{B^*} \leqslant C\|\hat{f}\|_B \tag{4·3}$$

for any $0 < \epsilon \leqslant 1$. The assumption (C2) implies

$$\|\hat{f}\|_B \leqslant C(\|f\|_B + \|f_\epsilon\|_B), \tag{4·4}$$

whilst assumption (C3) combined with relation (1·6) yields that, for $\alpha = \frac{1}{2} + \delta/4$,

$$\|f_\epsilon\|_B^2 \leqslant \|f_\epsilon\|_\alpha^2 \leqslant C\{\|u_\epsilon\|_{0,\Omega(R)}^2 + R^{-\delta}\|u_\epsilon\|_{-\alpha}^2\} \leqslant C\{\|v_\epsilon\|_{0,\Omega(R)}^2 + R^{-\delta}\|v_\epsilon\|_{B^*}^2\} \tag{4·5}$$

for sufficiently large $R > 0$, where C is independent of R and use has been made of the fact that $u_\epsilon = C_0 v_\epsilon$. Combining (4·3), (4·4) and (4·5) gives

$$\|u_\epsilon\|_{B^*} \leqslant C_R\{\|f\|_B + \|u_\epsilon\|_{0,\Omega(R)}\}. \tag{4·6}$$

Let $\psi \in C_0^\infty(\mathbf{R}^{n+1})$, $0 \leqslant \psi \leqslant 1$, $\psi = 0$ for $|z| \geqslant 2$ and $\psi = 1$ for $|z| \leqslant 1$. Set $\psi_R(z) = \psi(z/R)$. Then multiplying (1·8) by $\psi_R\bar{u}_\epsilon$ and integrating by parts over $\mathbf{R}^{n+1}$, we obtain on using Schwarz's inequality

$$\|\nabla u_\epsilon\|_{B^*}^2 \leqslant C_\sigma\{\|f\|_B^2 + \|u_\epsilon\|_{B^*}^2\} + \sigma\|\nabla u_\epsilon\|_{B^*}^2 \tag{4·7}$$

for arbitrarily small $\sigma > 0$. Take σ small enough to obtain

$$\|\nabla u_\epsilon\|_{B^*} \leqslant C\{\|f\|_B + \|u_\epsilon\|_{B^*}\}, \tag{4·8}$$

which together with (4·6) implies the estimate (4·1) for some $T \geqslant R$. The proof is complete.

We are now ready to prove Theorem 1·2. This will be done by removing the term

$\|u_\epsilon\|_{0,\Omega(T)}$ on the right-hand side of (4·1) using a compact argument combined with Theorem 1·1.

Proof of Theorem 1·2. To prove Theorem 1·2 it is sufficient to prove that there is a constant $C > 0$ such that for all $\epsilon \in (0, 1]$,

$$\|u_\epsilon\|_{0,\Omega(T)} \leqslant C\|f\|_B. \tag{4·9}$$

If (4·9) were false, then there would be a sequence $\{\epsilon_m\}_1^\infty$ such that $a_m \equiv \|u_{\epsilon_m}\|_{0,\Omega(T)}$ tends to infinity and ϵ_m tends to some τ with $0 \leqslant \tau \leqslant 1$ as $m \to \infty$. Set

$$v_m = u_{\epsilon_m}/a_m, \quad f_m = f/a_m.$$

Then v_m satisfies (1·8) with ϵ, f replaced by ϵ_m, f_m, respectively, and by Theorem 4·1

$$\|\nabla v_m\|_{B^*} + \|v_m\|_{B^*} \leqslant C\{\|f_m\|_B + 1\} < \infty. \tag{4·10}$$

Rellich's selection theorem along with (4·10) and standard elliptic estimates asserts that there is a subsequence of $\{v_m\}$, which is simply denoted by $\{v_m\}$, such that v_m strongly converges to some u in $H^2(\Omega(R))$ for all $R > 0$ as $m \to \infty$. From Lemma 1·2 and the fact that a sequence converging in $L_2(\Omega(R))$ for all $R > 0$ and bounded in $B^*(\mathbf{R}^{n+1})$ is Cauchy in any $L_2^{-\alpha}(\mathbf{R}^{n+1})$ with $\alpha > \frac{1}{2}$, it follows that v_m also strongly converges to u in $H^2_{-\alpha}(\mathbf{R}^{n+1})$ with $\alpha > \frac{1}{2}$ as $m \to \infty$. It is clear from Lemma 1·2 that it is enough to check the case when $\tau = 0$. We first notice that u satisfies (1·7) and

$$\|\nabla u\|_{B^*} + \|u\|_{B^*} \leqslant C < \infty. \tag{4·11}$$

Next, we show that u satisfies the radiation condition (1·5) as well. Then Theorem 1·1 yields that $u = 0$ almost everywhere in $\mathbf{R}^{n+1}$, which contradicts the fact that $\|v_m\|_{0,\Omega(T)} = 1$, and proves (4·9).

Since v_m satisfies (1·8), then it can be derived, in the same way as that used at the beginning of the proof of Theorem 4·1, that $v_m = C_0 G(\lambda + i\epsilon_m)\hat{f}_m$, where

$$\hat{f}_m = C_0\{\mu f_m + (\lambda + i\epsilon_m)(\mu - \mu_0)v_m\}.$$

By virtue of Theorem 2·3, (C3) and the relation (1·6), we find

$$G(\lambda + i\epsilon_m)\hat{f}_m \to G(\lambda + i0)\hat{f}$$

strongly in $L_2^{-\alpha}(\mathbf{R}^{n+1})$ with $\frac{1}{2} < \alpha \leqslant \frac{1}{2} + \delta/2$ as $m \to \infty$, where $\hat{f} = C_0\lambda(\mu - \mu_0)u$ and, by (4·11) and (C3), $\hat{f} \in L_2^{\alpha}(\mathbf{R}^{n+1}) \subset B(\mathbf{R}^{n+1})$. Consequently, we have

$$u = C_0 G(\lambda + i0)\hat{f}. \tag{4·12}$$

Using the defintion of v_m and $\hat{f}_m$ we obtain that

$$\Im(\hat{f}_m, G(\lambda + i\epsilon_m)\hat{f}_m) = \Im(\mu f_m, v_m) + \epsilon_m([\mu - \mu_0]v_m, v_m).$$

In view of (C3) and the fact that $\|v_m\|_{B^*}$ is uniformly bounded and v_m strongly converges to u in $H^2_{-\alpha}(\mathbf{R}^{n+1})$ with $\alpha > \frac{1}{2}$ as $m \to \infty$, the right-hand side of the above equation tends to zero as $m \to \infty$, whilst the left-hand side tends to $\Im(\hat{f}, G(\lambda + i0)\hat{f})$, so that $\Im(\hat{f}, G(\lambda+i0)\hat{f}) = 0$. Thus, by Proposition 2·1, $u = C_0 G(\lambda+i0)\hat{f} \in L_2^{-\beta}(\mathbf{R}^{n+1})$ for any $\beta > 0$, which implies that u satisfies (1·5). The theorem is thus proved.

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