“JUST THE MATHS”

SLIDES NUMBER

9.9

MATRICES 9
(Modal & spectral matrices)

by

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UNIT 9.9 - MATRICES 9

MODAL AND SPECTRAL MATRICES

9.9.1 ASSUMPTIONS AND DEFINITIONS

For convenience, we shall make, here, the following assumptions:

(a) The n eigenvalues, $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$, of an $n \times n$ matrix, A, are arranged in order of decreasing value.

(b) Corresponding to $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$ respectively, A possesses a full set of eigenvectors $\mathrm{X}_1,\ \mathrm{X}_2,\ \mathrm{X}_3, \ldots, \mathrm{X}_n$, which are **"linearly independent"**.

If two eigenvalues coincide, the order of writing down the corresponding pair of eigenvectors will be immaterial.

DEFINITION 1

The square matrix obtained by using, as its columns, any set of linearly independent eigenvectors of a matrix A is called a **"modal matrix"** of A, and may be denoted by M.

Notes:

(i) There are infinitely many modal matrices for a given matrix, A, since any multiple of an eigenvector is also an eigenvector.

(ii) It is sometimes convenient to use a set of normalised eigenvectors.

When using normalised eigenvectors, the modal matrix may be denoted by N and, for an $n \times n$ matrix, A, there are 2^n possibilities for N, since each of the n columns has two possibilities.

DEFINITION 2

If $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,\lambda_n$ are the eigenvalues of an $n \times n$ matrix, A, then the diagonal matrix,

$$\begin{bmatrix} \lambda_1 & 0 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & 0 \\ 0 & 0 & \lambda_3 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \lambda_n \end{bmatrix},$$

is called the **“spectral matrix”** of A, and may be denoted by S.

EXAMPLE

For the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

determine a modal matrix, a modal matrix of normalised eigenvectors and the spectral matrix.

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0,$$

which may be shown to give

$$-(1+\lambda)(1-\lambda)(2-\lambda) = 0.$$

Hence, the eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 1$ and $\lambda_3 = -1$ in order of decreasing value.

Case 1. $\lambda = 2$

We solve the simultaneous equations

$$\begin{aligned} -x + y - 2z &= 0, \\ -x + 0y + z &= 0, \\ 0x + y - 3z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 3 : 1$

Case 2. $\lambda = 1$

We solve the simultaneous equations

$$\begin{aligned} 0x + y - 2z &= 0, \\ -x + y + z &= 0, \\ 0x + y - 2z &= 0, \end{aligned}$$

which give $x : y : z = 3 : 2 : 1$

Case 3. $\lambda = -1$

We solve the simultaneous equations

$$\begin{aligned} 2x + y - 2z &= 0, \\ -x + 3y + z &= 0, \\ 0x + y + 0z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 0 : 1$

A modal matrix for A may therefore be given by

$$\mathrm{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

A modal matrix of normalised eigenvectors may be given by

$$\mathrm{N} = \begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{11}} & \frac{2}{\sqrt{14}} & 0 \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

9.9.2 DIAGONALISATION OF A MATRIX

Since the eigenvalues of a diagonal matrix are equal to its diagonal elements, it is clear that a matrix, A, and its spectral matrix, S, have the same eigenvalues.

THEOREM

The matrix, A, is similar to its spectral matrix, S, the similarity transformation being

$$M^{-1}AM = S,$$

where M is a modal matrix for A.

ILLUSTRATION:

Suppose that X_1, X_2 and X_3 are linearly independent eigenvectors of a 3×3 matrix, A, corresponding to eigenvalues λ_1, λ_2 and λ_3, respectively.

Then,

$$AX_1 = \lambda_1 X_1, \quad AX_2 = \lambda_2 X_2, \quad \text{and} \quad AX_3 = \lambda_3 X_3.$$

Also,

$$\mathrm{M} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}.$$

If M is premultiplied by A, we obtain a 3×3 matrix whose columns are AX_1, AX_2, and AX_3.

That is,

$$\mathrm{AM} = [\,\mathrm{AX}_1 \quad \mathrm{AX}_2 \quad \mathrm{AX}_3\,] = [\,\lambda_1 \mathrm{X}_1 \quad \lambda_2 \mathrm{X}_2 \quad \lambda_3 \mathrm{X}_3\,]$$

or

$$\mathrm{AM} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} . \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathrm{MS}.$$

We conclude that

$$\mathrm{M}^{-1}\mathrm{AM} = \mathrm{S}.$$

Notes:

(i) M^{-1} exists only because X_1, X_2 and X_3 are linearly independent.

(ii) The similarity transformation in the above theorem reduces the matrix, A, to **"diagonal form"** or **"canonical form"** and the process is often referred to as the **"diagonalisation"** of the matrix, A.

EXAMPLE

Verify the above Theorem for the matrix

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Solution

From an earlier example, a modal matrix for A may be given by

$$\mathrm{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The spectral matrix is given by

$$\mathrm{S} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It may be shown that

$$\mathrm{M}^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix}$$

and, hence,

$$\mathrm{M}^{-1}\mathrm{AM} = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} . \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix} . \begin{bmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathrm{S}.$$