

**“JUST THE MATHS”**

**SLIDES NUMBER**

**9.5**

**MATRICES 5**  
**(Consistency and rank)**

by

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**9.5.1 The consistency of simultaneous linear equations**

**9.5.2 The row-echelon form of a matrix**

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## UNIT 9.5 - MATRICES 5

### CONSISTENCY AND RANK

#### 9.5.1 THE CONSISTENCY OF SIMULTANEOUS LINEAR EQUATIONS

##### Introduction

Some sets of equations cannot be solved to give a unique solution.

The Gaussian Elimination method is able to detect such sets.

##### ILLUSTRATION 1.

$$\begin{aligned}3x - y + z &= 1, \\2x + 2y - 5z &= 0, \\5x + y - 4z &= 7.\end{aligned}$$

Gaussian Elimination gives

$$\begin{array}{ccc|c|c}3 & -1 & 1 & 1 & 4 \\2 & 2 & -5 & 0 & -1 \\5 & 1 & -4 & 7 & 9\end{array}$$

$$R_1 \rightarrow R_1 - R_2$$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 7 & 9 \end{array}$$

$$R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 5R_1$$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 16 & -34 & 2 & -16 \end{array}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 0 & 0 & 6 & 6 \end{array}$$

The third line seems to imply that  $0 \cdot z = 6$ ; that is,  $0 = 6$  which is impossible.

Hence the equations have no solution and are said to be “**inconsistent**”.

## **ILLUSTRATION 2.**

$$\begin{aligned} 3x - y + z &= 1, \\ 2x + 2y - 5z &= 0, \\ 5x + y - 4z &= 1. \end{aligned}$$

Gaussian Elimination gives

$$\begin{array}{ccc|c|c} 3 & -1 & 1 & 1 & 4 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 1 & 3 \end{array}$$

$$R_1 \rightarrow R_1 - R_2$$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 1 & 3 \end{array}$$

$$R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 5R_1$$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 16 & -34 & -4 & -22 \end{array}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

Hence, the original set of equations is equivalent to

$$\begin{aligned} x - 3y + 6z &= 1, \\ 8y - 17z &= -2. \end{aligned}$$

## First form of solution

For the two equations

$$\begin{aligned}x - 3y + 6z &= 1, \\8y - 17z &= -2.\end{aligned}$$

Any one of the three variables may be chosen at random, the other two being expressed in terms of it.

For instance, if we choose  $z$  at random, then

$$y = \frac{17z - 2}{8} \text{ giving } x = \frac{3z + 2}{8}.$$

## Neater form of solution

Let  $x = x_0$ ,  $y = y_0$ ,  $z = z_0$  be any **known** solution. For example, if  $z = 0$ ,  $y = -\frac{1}{4}$  and hence  $x = \frac{1}{4}$ .

Let us now substitute

$$\begin{aligned}x &= x_1 + x_0, \\y &= y_1 + y_0, \\z &= z_1 + z_0.\end{aligned}$$

We obtain

$$\begin{aligned}(x_1 + x_0) - 3(y_1 + y_0) + 6(z_1 + z_0) &= 1, \\ 8(y_1 + y_0) - 17(z_1 + z_0) &= -2.\end{aligned}$$

Because  $(x_0, y_0, z_0)$  is a known solution, this reduces to

$$\begin{aligned}x_1 - 3y_1 + 6z_1 &= 0, \\ 8y_1 - 17z_1 &= 0.\end{aligned}$$

This is a set of **“homogeneous linear equations”**.

Although clearly satisfied by  $x_1 = 0$ ,  $y_1 = 0$ ,  $z_1 = 0$ , we regard this as a **“trivial solution”** and ignore it.

An infinite number of non-trivial solutions can be found for each of which the variables  $x_1$ ,  $y_1$  and  $z_1$  are in a certain set of ratios.

From

$$\begin{aligned}x_1 - 3y_1 + 6z_1 &= 0, \\8y_1 - 17z_1 &= 0,\end{aligned}$$

the second equation gives

$$y_1 = \frac{17}{8}z_1 \text{ which means that } y_1 : z_1 = 17 : 8.$$

Substituting into the first equation,

$$x_1 - \frac{51}{8}z_1 + 6z_1 = 0 \text{ which means that } x_1 = \frac{3}{8}z_1 \text{ or } x_1 : z_1 = 3 : 8.$$

Combining these conclusions

$$x_1 : y_1 : z_1 = 3 : 17 : 8.$$

Any three numbers in these ratios will serve as a set of values for  $x_1$ ,  $y_1$  and  $z_1$ .

In general, the neater form of solution can be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \alpha \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},$$

where  $\alpha$  may be any non-zero number.

In our present example,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ 17 \\ 8 \end{bmatrix}.$$

**Notes:**

(i) Since at least one of the variables is going to be non-zero, we may begin by letting it equal 1.

In the equations

$$\begin{aligned} x_1 - 3y_1 + 6z_1 &= 0, \\ 8y_1 - 17z_1 &= 0, \end{aligned}$$

suppose we let  $z_1 = 1$ .

Then

$$y_1 = \frac{17}{8} \quad \text{and} \quad x_1 = \frac{3}{8}.$$

Hence,

$$x_1 : y_1 : z_1 = \frac{3}{8} : \frac{17}{8} : 1,$$

which can be rewritten

$$x_1 : y_1 : z_1 = 3 : 17 : 8.$$

(ii) It may happen that a set of simultaneous linear equations reduces to **only one** equation.

That is, each equation is a multiple of the first

### **ILLUSTRATION 3.**

Suppose a set of equations reduces to

$$3x - 2y + 5z = 6.$$

A particular solution is  $x_0 = 1, y_0 = 1, z_0 = 1$ .

The general solution is given by

$$x = x_0 + x_1, \quad y = y_0 + y_1, \quad z = z_0 + z_1,$$

where

$$3x_1 - 2y_1 + 5z_1 = 0.$$

Let  $x_1 = \alpha$  and  $y_1 = \beta$  (random numbers).

Hence,

$$z_1 = \frac{2\beta - 3\alpha}{5}.$$

Using  $x = \alpha$ ,  $y = \beta$  and  $z = \frac{2\beta-3\alpha}{5}$ , we may write

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

## 9.5.2 THE ROW-ECHELON FORM OF A MATRIX

Using Gaussian Elimination to solve

$$MX = K,$$

we begin with the augmented matrix  $M|K$ .

Then we use elementary row operations to obtain more zeros at the beginning of each row than at the beginning of the previous row.

If desired, the first non-zero element in each row could be reduced to 1 on dividing that row throughout by the value of the first non-zero element.

## DEFINITION

The “row echelon form” of a matrix is that for which the first non-zero element in each row is 1 and occurs to the right of the first non-zero element in the previous row.

### Note:

In practice, for “row echelon form”, the first non-zero element in each row need not necessarily be reduced to 1.

## 9.5.3 THE RANK OF A MATRIX

### DEFINITION

The “**rank** ” of a matrix is the number of rows which do not reduce to a complete row of zeros when the matrix has been converted to row echelon form.

### ILLUSTRATIONS

1. In the previous Illustration 1,  $M$  had rank 2 but  $M|K$  had rank 3. The equations were inconsistent.
2. In the previous Illustration 2,  $M$  had rank 2 and  $M|K$  also had rank 2. The equations had an infinite number of solutions.

3. Whenever both  $M$  and  $M|K$  have rank 3, there is a unique solution to the simultaneous equations.

## SUMMARY

1. The equations  $MX = K$  are inconsistent if

$$\text{rank } M < \text{rank } M|K.$$

2. The equations  $MX = K$  have an infinite number of solutions if

$$\text{rank } M = \text{rank } M|K < n,$$

where  $n$  is the number of equations.

3. The equations  $MX = K$  have a unique solution if

$$\text{rank } M = \text{rank } M|K = n,$$

where  $n$  is the number of equations.