

“JUST THE MATHS”

SLIDES NUMBER

9.4

**MATRICES 4
(Row operations)**

by

A.J.Hobson

9.4.1 Matrix inverses by row operations

9.4.2 Gaussian elimination (the elementary version)

UNIT 9.4 - MATRICES 4

ROW OPERATIONS

9.4.1 MATRIX INVERSES BY ROW OPERATIONS

DEFINITION

An “**elementary row operation**” on a matrix is any one of the following three possibilities:

- (a) The interchange of two rows;
- (b) The multiplication of the elements in any row by a non-zero number;
- (c) The addition of the elements in any row to the corresponding elements in another row.

Note:

From types (b) and (c), the elements in any row may be **subtracted** from the corresponding elements in another row.

More generally, **multiples** of the elements in any row may be added to or subtracted from the corresponding elements in another row.

RESULT 1.

To perform an elementary row operation on a matrix **algebraically**, we may pre-multiply the matrix by an identity matrix on which the same elementary row operation has been already performed.

For example, in the matrix

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix},$$

subtract twice the third row from the second row.

This could be regarded as the succession of two elementary row operations as follows:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ 2a_3 & 2b_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 - 2a_3 & b_2 - 2b_3 \\ a_3 & b_3 \end{bmatrix}. \end{aligned}$$

DEFINITION

An “**elementary matrix**” is a matrix obtained from an identity matrix by performing upon it one elementary row operation.

RESULT 2.

If a certain sequence of elementary row operations converts a given square matrix, M , into the corresponding identity matrix, then the same sequence of elementary row operations in the same order will convert the identity matrix into M^{-1} .

Proof:

Suppose that

$$E_n \cdot E_{n-1} \dots E_4 \cdot E_3 \cdot E_2 \cdot E_1 \cdot M = I,$$

where $E_1, E_2, E_3, E_4, \dots, E_{n-1}, E_n$ are elementary matrices.

Post-multiplying both sides with M^{-1} ,

$$E_n \cdot E_{n-1} \dots E_4 \cdot E_3 \cdot E_2 \cdot E_1 \cdot M \cdot M^{-1} = I \cdot M^{-1}.$$

In other words,

$$E_n \cdot E_{n-1} \dots E_4 \cdot E_3 \cdot E_2 \cdot E_1 \cdot I = M^{-1},$$

which proves the result.

EXAMPLES

1. Use elementary row operations to determine the inverse of the matrix

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}.$$

Solution

First we write down the given matrix side-by-side with the corresponding identity matrix.

$$\begin{bmatrix} 3 & 7 & : & 1 & 0 \\ 2 & 5 & : & 0 & 1 \end{bmatrix}.$$

Secondly, we try to arrange that the first element in the first column of this arrangement is 1.

This can be carried out by subtracting the second row from the first row.

(Instruction: $R_1 \rightarrow R_1 - R_2$).

$$\begin{bmatrix} 1 & 2 & : & 1 & -1 \\ 2 & 5 & : & 0 & 1 \end{bmatrix}.$$

Thirdly, we try to convert the first column of the display into the first column of the identity matrix

This can be carried out by subtracting twice the first row from the second row.

(Instruction: $R_2 \rightarrow R_2 - 2R_1$).

$$\begin{bmatrix} 1 & 2 & : & 1 & -1 \\ 0 & 1 & : & -2 & 3 \end{bmatrix}.$$

Lastly, we try to convert the second column of the display into the second column of the identity matrix.

This can be carried out by subtracting twice the second row from the first row

(Instruction: $R_1 \rightarrow R_1 - 2R_2$).

$$\begin{bmatrix} 1 & 0 & : & 5 & -7 \\ 0 & 1 & : & -2 & 3 \end{bmatrix}.$$

The inverse matrix is therefore

$$\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}.$$

Notes:

(i) The technique used for a 2×2 matrix applies to square matrices of all orders with appropriate modifications.

(ii) The idea is to obtain, in chronological order, the columns of the identity matrix from the columns of the given matrix.

We do this by converting each diagonal element in the given matrix to 1 and then using multiples of 1 to reduce the remaining elements in the same column to zero.

- (iii) Once any row has been used to reduce elements to zero, that row must not be used again as the operator;
2. Use elementary row operations to show that the matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

has no inverse.

Solution

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array} \right];$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array} \right].$$

There is no way now of continuing to convert the given matrix into the identity matrix; hence, there is no inverse.

3. Use elementary row operations to find the inverse of the matrix

$$\begin{bmatrix} 4 & 1 & 6 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}.$$

Solution

$$\left[\begin{array}{ccc|ccc} 4 & 1 & 6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array} \right];$$

$$R_1 \rightarrow R_1 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array} \right];$$

$$R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & -2 & 1 & 2 \\ 0 & 5 & 2 & -3 & 0 & 4 \end{array} \right];$$

$$R_2 \rightarrow 2R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & -3 & 0 & 4 \end{array} \right];$$

$$R_1 \rightarrow R_1 + R_2 \text{ and } R_3 \rightarrow R_3 - 5R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 2 & -1 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 2 & \vdots & 2 & -10 & 4 \end{bmatrix};$$

$$R_3 \rightarrow R_3 \times \frac{1}{2}$$

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 2 & -1 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 1 & \vdots & 1 & -5 & 2 \end{bmatrix};$$

$$R_1 \rightarrow R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & -1 & 7 & -3 \\ 0 & 1 & 0 & \vdots & -1 & 2 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 5 & 2 \end{bmatrix}.$$

The required inverse matrix is therefore

$$\begin{bmatrix} -1 & 7 & -3 \\ -1 & 2 & 0 \\ 1 & -5 & 2 \end{bmatrix}.$$

9.4.2 GAUSSIAN ELIMINATION THE ELEMENTARY VERSION

The method will be introduced through the case of three equations in three unknowns, but may be applied in other cases as well.

First, consider the special equations

$$\begin{aligned}a_1x + b_1y + c_1z &= k_1, \\b_2y + c_2z &= k_2, \\c_3z &= k_3.\end{aligned}$$

We may obtain z from the third equation.

Substituting for z into the second equation, we may find y .

Substituting for both y and z in the first equation we may find x .

The method of Gaussian Elimination reduces any set of linear equations to this triangular form by adding or subtracting suitable multiples of pairs of the equations.

The method is usually laid out in a tabular form using only the coefficients of the variables and the constant terms.

EXAMPLE

Solve the simultaneous linear equations

$$2x + y + z = 3,$$

$$x - 2y - z = 2,$$

$$3x - y + z = 8.$$

Solution

We try to arrange that the first coefficient in the first equation is 1.

Here, we could interchange the first two equations.

$$\begin{array}{ccc|c} \boxed{1} & -2 & -1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -1 & 1 & 8 \end{array}$$

This format is known as an “**augmented matrix**”. The matrix of coefficients has been augmented by the matrix of constant terms.

Using $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ gives a new table:

$$\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & \boxed{5} & 3 & -1 \\ 0 & 5 & 4 & 2 \end{array}$$

$R_3 \rightarrow R_3 - R_2$ gives another new table:

$$\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & 1 & 3 \end{array}$$

The numbers enclosed in the boxes are called the “**pivot elements**” and are used to reduce to zero the elements below them in the same column.

The final table provides a new set of equations, equivalent to the original:

$$\begin{aligned} x - 2y - z &= 2 \\ 5y + 3z &= -1 \\ z &= 3 \end{aligned}$$

Hence, $\boxed{z = 3, y = -2, x = 1}$.

INSERTING A CHECK COLUMN

With large numbers of equations, often involving awkward decimal quantities, the margin for error is greatly increased.

As a check on the arithmetic at each stage, we introduce some **additional** arithmetic which has to remain consistent with the calculations already being carried out.

Add together the numbers in each row in order to produce an extra column.

Each row operation is then performed on the extended rows.

In the new table, the final column should still be the sum of the numbers to the left of it.

The working for the previous example would be as follows:

$$\begin{array}{ccc|c|c} \boxed{1} & -2 & -1 & 2 & 0 \\ 2 & 1 & 1 & 3 & 7 \\ 3 & -1 & 1 & 8 & 11, \end{array} \quad \begin{array}{ccc|c|c} 1 & -2 & -1 & 2 & 0 \\ 0 & \boxed{5} & 3 & -1 & 7 \\ 0 & 5 & 4 & 2 & 11, \end{array} \quad \begin{array}{ccc|c|c} 1 & -2 & -1 & 2 & 0 \\ 0 & 5 & 3 & -1 & 7 \\ 0 & 0 & 1 & 3 & 4 \end{array}$$

Using $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$
and $R_3 \rightarrow R_3 - R_2$.