"JUST THE MATHS"

SLIDES NUMBER

9.3

MATRICES 3
(Matrix inversion & simultaneous equations)

by

A.J.Hobson

UNIT 9.3 - MATRICES 3

MATRIX INVERSION AND SIMULTANEOUS EQUATIONS

9.3.1 INTRODUCTION

In Matrix Algebra, there is no such thing as **division** in the usual sense.

An equivalent operation called **"inversion"** is similar to the process where division by a value, a, is the same as multiplication by $\frac{1}{a}$.

For example, consider the equation

$$mx = k.$$

The solution is obviously

$$x = \frac{k}{m}.$$

Alternatively,

(a) Pre-multiply both sides of the given equation by m^{-1}

$$m^{-1}.(mx) = m^{-1}k.$$

(b) Rearrange this as

$$(m^{-1}.m)x = m^{-1}k.$$

(c) Use $m^{-1}.m = 1$ to give

$$1.x = m^{-1}k.$$

(d) Use $1.x = x$ to give

$$x = m^{-1}k.$$

Later, we see an almost identical sequence of steps, with matrices

Matrix inversion is developed from the rules for matrix multiplication.

9.3.2 MATRIX REPRESENTATION OF SIMULTANEOUS LINEAR EQUATIONS

In this section, we consider three simultaneous linear equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3. \end{aligned}$$

These can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

or

$$\mathrm{MX} = \mathrm{K}.$$

Note:
Suppose N is such that NM = I.

Pre-multiply MX = K by N to give

$$\mathrm{N(MX)} = \mathrm{NK}.$$

That is,

$$\mathrm{(NM)X} = \mathrm{NK}.$$

In other words,

$$\mathrm{IX} = \mathrm{NK}.$$

Hence,

$$\mathrm{X} = \mathrm{NK}.$$

N exhibits a similar behaviour to the number m^{-1} encountered earlier; we replace N with M^{-1}.

9.3.3 THE DEFINITION OF A MULTIPLICATIVE INVERSE

The **"multiplicative inverse"** of a square matrix M is another matrix, denoted by M^{-1} which has the property

$$M^{-1}.M = I.$$

Notes:

(i) It is certainly **possible** for the product of two matrices to be an identity matrix
(see Unit 9.2, Exercises)

(ii) We may usually call M^{-1} the "inverse" of M rather than the "multiplicative inverse".

(iii) It can be shown that, when $M^{-1}.M = I$, it is also true that

$$M.M^{-1} = I.$$

(iv) A square matrix cannot have more than one inverse.

Assume that A had two inverses, B and C.

Then,

$$C = CI = C(AB) = (CA)B = IB = B.$$

9.3.4 THE FORMULA FOR A MULTIPLICATIVE INVERSE

(a) The inverse of a 2 x 2 matrix

Taking

$$\mathrm{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

and

$$\mathrm{M}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

we require that

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} . \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} a_1P + b_1R &= 1, \\ a_2P + b_2R &= 0, \\ a_1Q + b_1S &= 0, \\ a_2Q + b_2S &= 1. \end{aligned}$$

These equations are satisfied by

$$P = \frac{b_2}{|\mathrm{M}|}, \quad Q = -\frac{b_1}{|\mathrm{M}|}, \quad R = -\frac{a_2}{|\mathrm{M}|} \quad S = \frac{a_1}{|\mathrm{M}|},$$

where

$$|\mathrm{M}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

$|\mathrm{M}|$ is called **"the determinant of the matrix** M".

Summary

$$\mathrm{M}^{-1} = \frac{1}{|\mathrm{M}|} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

EXAMPLES

1. Write down the inverse of the matrix

$$M = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}.$$

Solution

$$| M |= 41.$$

Hence,

$$M^{-1} = \frac{1}{41} \begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix}.$$

Check

$$M^{-1}.M = \frac{1}{41} \begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix} . \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix} = \frac{1}{41} \begin{bmatrix} 41 & 0 \\ 0 & 41 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Use matrices to solve the simultaneous linear equations

$$\begin{aligned} 3x + y &= 1, \\ x - 2y &= 5. \end{aligned}$$

Solution

The equations can be written MX = K, where

$$\mathrm{M} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathrm{X} = \begin{bmatrix} x \\ y \end{bmatrix} \ \text{ and } \ \mathrm{K} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

First, check that |M| $\neq$ 0.

$$\mid \mathrm{M} \mid = \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -6 - 1 = -7.$$

Thus,

$$\mathrm{M}^{-1} = -\frac{1}{7}\begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix}.$$

The solution of the simultaneous equations is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -7 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

That is,

$$x = 1 \quad y = -2.$$

(b) The inverse of a 3 x 3 Matrix

We use another version of Cramer's rule.

The simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

have the solution

$$\frac{x}{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}$$

$$= \frac{z}{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

METHOD

(i) The last determinant above is called **"the determinant of the matrix** M**"**, denoted by | M |.

In |M|, we let A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3 denote the **"cofactors"** (or **"signed minors"**) of a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2 and c_3 respectively.

(ii) For each of k_1, k_2 and k_3 the cofactor is the same as the corresponding cofactor in | M |.

(iii) The solutions for x, y and z can be written as follows:

$$x = \frac{1}{|\,\mathrm{M}\,|}\left(k_1A_1 + k_2A_2 + k_3A_3\right);$$

$$y = \frac{1}{|\,\mathrm{M}\,|}\left(k_1B_1 + k_2B_2 + k_3B_3\right);$$

$$z = \frac{1}{|\,\mathrm{M}\,|}\left(k_1C_1 + k_2C_2 + k_3C_3\right);$$

or, in matrix format,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|\,\mathrm{M}\,|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

Compare this with

$$\mathrm{X} = \mathrm{M}^{-1}K.$$

We conclude that

$$\mathrm{M}^{-1} = \frac{1}{\mid \mathrm{M} \mid} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

Summary
Similar working would occur for larger or smaller systems of equations.

In general, the inverse of a square matrix is **the transpose of the matrix of cofactors times the reciprocal of the determinant of the matrix**.

Notes:
(i) If $\mid \mathrm{M} \mid = 0$, then the matrix M does not have an inverse and is said to be **"singular"**.

If $|\mathrm{M}| \neq 0$, M is said to be **"non-singular"**.

(ii) The transpose of the matrix of cofactors is called the **"adjoint"** of M, denoted by AdjM.

There is always an adjoint though not always an inverse.

When the inverse exists,

$$\mathrm{M}^{-1} = \frac{1}{\mid \mathrm{M} \mid}\mathrm{AdjM}.$$

(iii) The inverse of a matrix of order 2×2 fits the above scheme also.

The cofactor of each element will be a single number associated with a **"place-sign"** according to the following pattern:

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}.$$

Hence, if

$$\mathrm{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix},$$

then,

$$\mathrm{M}^{-1} = \frac{1}{a_1b_2 - a_2b_1} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

The matrix part of the result can be obtained by interchanging the diagonal elements of M and reversing the signs of the other two elements.

EXAMPLE

Use matrices to solve the simultaneous linear equations

$$
\begin{aligned}
3x + y - z &= 1, \\
x - 2y + z &= 0, \\
2x + 2y + z &= 13.
\end{aligned}
$$

Solution

The equations can be written MX = K, where

$$
\mathrm{M} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad \mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathrm{K} = \begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix}.
$$

$$
\mid \mathrm{M} \mid = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 3(-2-2)-1(1-2)+(-1)(2+4).
$$

$$
|\mathrm{M}| = -17.
$$

If C denotes the matrix of cofactors, then

$$\mathrm{C} = \begin{bmatrix} \boxed{-4} & 1 & \boxed{6} \\ -3 & \boxed{5} & -4 \\ \boxed{-1} & -4 & \boxed{-7} \end{bmatrix}.$$

Notes:

(i) The framed elements indicate those for which the place sign is positive.

(ii) The remaining four elements are those for which the place sign is negative.

(iii) In finding the elements of C, **do not multiply the cofactors of the elements in** M **by the elements themselves**.

The Inverse is given by

$$\mathrm{M}^{-1} = \frac{1}{|\mathrm{M}|}\mathrm{AdjM} = \frac{1}{-17}C^{\mathrm{T}} = \frac{1}{-17}\begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix}.$$

The solution of the equations is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -17 \\ -51 \\ -85 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$