

“JUST THE MATHS”

SLIDES NUMBER

9.2

MATRICES 2
(Further matrix algebra)

by

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- 9.2.1 Multiplication by a single number
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UNIT 9.2 - MATRICES 2

THE ALGEBRA OF MATRICES (Part Two)

9.2.1 MULTIPLICATION BY A SINGLE NUMBER

Multiplying a matrix of any order by a **positive whole number**, n , is equivalent to adding together n copies of the given matrix.

Every element would be multiplied by n .

To extend this idea to multiplication by **any** number, λ , we multiply every element by λ .

In symbols,

$$\lambda [a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$$

where

$$b_{ij} = \lambda a_{ij}$$

Note:

The rule for multiplying a matrix by a single number can also be used to remove common factors from the elements of a matrix.

ILLUSTRATION

$$\begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

9.2.2 THE PRODUCT OF TWO MATRICES

We introduce the definition with a semi-practical illustration.

ILLUSTRATION

A motor manufacturer, with three separate factories, makes two types of car - one called “standard” and the other called “luxury”.

In order to manufacture each type of car, he needs a certain number of units of material and a certain number of units of labour each unit representing £300.

A table of data to represent this information could be

Type	Materials	Labour
Standard	12	15
Luxury	16	20

The manufacturer receives an order from another country to supply 400 standard cars and 900 luxury cars.

He distributes the export order as follows:

Location	Standard	Luxury
Factory A	100	400
Factory B	200	200
Factory C	100	300

The number of units of material and labour needed to complete the order may be given by the following table:

Location	Materials	Labour
Factory A	$100 \times 12 + 400 \times 16$	$100 \times 15 + 400 \times 20$
Factory B	$200 \times 12 + 200 \times 16$	$200 \times 15 + 200 \times 20$
Factory C	$100 \times 12 + 300 \times 16$	$100 \times 15 + 300 \times 20$

In matrix notation,

$$\begin{bmatrix} 100 & 400 \\ 200 & 200 \\ 100 & 300 \end{bmatrix} \cdot \begin{bmatrix} 12 & 15 \\ 16 & 20 \end{bmatrix} =$$

$$\begin{bmatrix} 100 \times 12 + 400 \times 16 & 100 \times 15 + 400 \times 20 \\ 200 \times 12 + 200 \times 16 & 200 \times 15 + 200 \times 20 \\ 100 \times 12 + 300 \times 16 & 100 \times 15 + 300 \times 20 \end{bmatrix} =$$

$$\begin{bmatrix} 7600 & 9500 \\ 5600 & 7000 \\ 6000 & 7500 \end{bmatrix}.$$

OBSERVATIONS

- (i) The product matrix has 3 rows because the first matrix on the left has 3 rows.
- (ii) The product matrix has 2 columns because the second matrix on the left has 2 columns.
- (iii) The product cannot be worked out unless the number of columns in the first matrix matches the number of rows in the second matrix.
- (iv) The elements of the product matrix are systematically obtained by multiplying (in pairs) the corresponding elements of each row in the first matrix with each column in the second matrix.

We read each row of the first matrix from left to right and each column of the second matrix from top to bottom.

The Formal Definition of a Matrix Product

If A and B are matrices, then the product AB is defined (that is, it has a meaning) only when the number of columns in A is equal to the number of rows in B .

If A is of order $m \times n$ and B is of order $n \times p$, then AB is of order $m \times p$.

To obtain the element in the i -th row and j -th column of AB , we multiply corresponding elements of the i -th row of A and the j -th column of B then add up the results.

ILLUSTRATION

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 1 & -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & -1 & 12 \\ 1 & -10 & 1 & 15 \end{bmatrix}$$

Note:

A matrix of any order can be multiplied by a matrix of order 1×1 since this is taken as a single number.

9.2.3 THE NON-COMMUTATIVITY OF MATRIX PRODUCTS

In elementary arithmetic, if a and b are two numbers, then $ab = ba$ (that is, the product “**commutes**”).

This is not so for matrices A and B .

(a) If A is of order $m \times n$, then B must be of order $n \times m$ if both AB **and** BA are to be defined.

(b) AB and BA will have different orders unless $m = n$, in which case the two products will be square matrices of order $m \times m$.

(c) Even if A and B are **both** square matrices of order $m \times m$, it will not normally be the case that AB is the same as BA .

EXAMPLE

$$\begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 11 & 35 \end{bmatrix};$$

but,

$$\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 19 & 37 \end{bmatrix}.$$

Notes:

(i) To show only that $AB \neq BA$, we simply demonstrate that one pair of corresponding elements are unequal in value.

(ii) It turns out that the non-commutativity of matrix products is the only algebraic rule which causes problems.

Others are O.K. such as

$$A + B \equiv B + A;$$

the **“Commutative Law of Addition”**.

$$A + (B + C) \equiv (A + B) + C;$$

the **“Associative Law of Addition”**.

$$A(BC) \equiv (AB)C;$$

the **“Associative Law of Multiplication”**.

$$A(B + C) \equiv AB + AC \text{ or } (A + B)C \equiv AC + BC;$$

the **“Distributive Laws”**.

(iii) In the matrix product, AB , we say either that B is **“pre-multiplied”** by A or that A is **“post-multiplied”** by B .

9.2.4 MULTIPLICATIVE IDENTITY MATRICES

A square matrix with 1's on the leading diagonal and zeros elsewhere is called a “**multiplicative identity matrix**”.

An $n \times n$ multiplicative identity matrix is denoted by I_n .

Sometimes, the notation I , without a subscript, is sufficient.

EXAMPLES

$$I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I_4 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

I_n multiplies another matrix (with an appropriate number of rows or columns) to leave it unchanged.

We use just “identity matrix” (unless it is necessary to distinguish it from the **additive** identity matrix referred to earlier).

Another common name for an identity matrix is a “**unit matrix**”.

ILLUSTRATION

Suppose that

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Then, post-multiplying by I_2 , it is easily checked that

$$AI_2 = A.$$

Similarly, pre-multiplying by I_3 , it is easily checked that

$$I_3A = A.$$

In general, if A is of order $m \times n$, then

$$AI_n = I_mA = A.$$