

“JUST THE MATHS”

SLIDES NUMBER

9.1

MATRICES 1

(Definitions & elementary matrix algebra)

by

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9.1.1 Introduction

9.1.2 Definitions

9.1.3 The algebra of matrices (part one)

UNIT 9.1 - MATRICES 1 - DEFINITIONS AND ELEMENTARY MATRIX ALGEBRA

9.1.1 INTRODUCTION

(a) Presentation of Data

Sets of numerical information can often be presented as a rectangular “**array**” of numbers.

For example, a football results table:

TEAM	PLAYED	WON	DRAWN	LOST
Blackburn	22	11	6	5
Burnley	22	9	6	7
Chelsea	21	7	8	6
Leicester	21	6	8	7
Stoke	21	6	6	9

If the headings are taken for granted, we write simply

$$\begin{bmatrix} 22 & 11 & 6 & 5 \\ 22 & 9 & 6 & 7 \\ 21 & 7 & 8 & 6 \\ 21 & 6 & 8 & 7 \\ 21 & 6 & 6 & 9 \end{bmatrix}$$

This symbol is called a “**matrix**”.

(b) Presentation of Algebraic Results

In two-dimensional geometry, the “**vector**” \underline{OP}_0 can be moved to the position \underline{OP}_1 by means of a “**reflection**”, a “**rotation**”, a “**magnification**” or a combination of such operations.

The relationship between the co-ordinates (x_0, y_0) of P_0 and (x_1, y_1) of P_1 is given by

$$\begin{aligned}x_1 &= ax_0 + by_0, \\y_1 &= cx_0 + dy_0.\end{aligned}$$

ILLUSTRATIONS

1. The equations

$$\begin{aligned}x_1 &= -x_0, \\y_1 &= y_0\end{aligned}$$

represent a reflection in the y -axis.

2. The equations

$$\begin{aligned}x_1 &= kx_0, \\y_1 &= ky_0\end{aligned}$$

represent a magnification when $|k| > 1$ and a contraction when $|k| < 1$.

3. The equations

$$\begin{aligned}x_1 &= x_0 \cos \theta - y_0 \sin \theta, \\y_1 &= x_0 \sin \theta + y_0 \cos \theta\end{aligned}$$

represent a rotation of \underline{OP}_0 through an angle θ in a counter-clockwise direction.

Such “**linear transformations**” are completely specified by a, b, c, d in the correct positions.

When referring to a linear transformation, we write the matrix symbol

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

To show that a transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

operates on the vector \underline{OP}_0 , transforming it into the vector \underline{OP}_1 , we write

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

9.1.2 DEFINITIONS

(i) A matrix is a “**rectangular array of numbers**” arranged in rows (horizontally), columns (vertically) and enclosed in brackets.

ILLUSTRATIONS:

$$\begin{bmatrix} 1 & 3 & 7 \\ 9 & 9 & 10 \end{bmatrix}, \begin{bmatrix} 10 & 16 & 17 \\ 3 & 5 & 11 \\ 4 & 0 & 10 \end{bmatrix}, [1 \quad 5 \quad 7].$$

Note:

Rows are counted from top to bottom and columns are counted from left to right.

(ii) Any number within the array is called an “**element**” of the matrix.

The $ij - th$ element is the element lying in the i -th row and the j -th column of the matrix.

(iii) If a matrix has m rows and n columns, it is called a “**matrix of order $m \times n$** ” or simply an “ **$m \times n$ matrix**”. It clearly has mn elements.

(iv) A matrix of order $m \times m$ is called a “**square**” matrix.

Note:

A matrix of order 1×1 is considered to be the same as a single number.

(v) A matrix of order $m \times 1$ is called a “**column vector**” and a matrix of order $1 \times n$ is called a “**row vector**”.

(vi) An arbitrary matrix whose elements and order do not have to be specified may be denoted by a single capital letter such as A,B,C, etc.

An arbitrary matrix of order $m \times n$ may be denoted fully by the symbol

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In each double-subscript, the first number is the row number and the second is the column number.

If the matrix is square, the elements $a_{11}, a_{22}, a_{33}, \dots, a_{mm}$ form the “**leading diagonal**”.

They are called the “**diagonal elements**” and their sum is called the “**trace**” of the matrix.

An abbreviated form is $[a_{ij}]_{m \times n}$.

EXAMPLE

A matrix $A = [a_{ij}]_{2 \times 3}$ is such that $a_{ij} = i^2 + 2j$. Write out A in full.

Solution

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 6 & 8 & 10 \end{bmatrix}$$

.

(vii) Given a matrix, A , of order $m \times n$, the matrix of order $n \times m$ obtained from A by writing the rows as columns is called the “**transpose**” of A .

The transpose is denoted by A^T - some books use A' or \tilde{A} .

ILLUSTRATION

If

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix},$$

then,

$$A^T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

(viii) A matrix, A , is said to be “**symmetric**” if $A = A^T$.

ILLUSTRATION

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

is symmetric.

(ix) A matrix, A , is said to be “**skew-symmetric**” if the elements of A^T are minus the corresponding elements of A itself. This will mean that the leading diagonal elements must be zero.

ILLUSTRATION

$$A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

is skew-symmetric.

(x) A matrix is said to be “**diagonal**” if the elements which are not on the leading diagonal have value zero while the elements on the leading diagonal are not all equal to zero.

9.1.3 THE ALGEBRA OF MATRICES (Part One)

1. Equality

Two matrices are said to be equal if they have the same order and also pairs of elements in corresponding positions are equal in value.

In symbols,

$$[a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$$

provided

$$a_{ij} = b_{ij}.$$

ILLUSTRATION

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

implies that $x = 1$, $y = 2$ and $z = 3$.

2. Addition and Subtraction

The sum and difference of two matrices are defined only when they have the same order.

The sum of two matrices of order $m \times n$ is formed by adding together the pairs of elements in corresponding positions.

The difference of two matrices of order $m \times n$ is formed by subtracting the pairs of elements in corresponding positions.

In symbols,

$$[a_{ij}]_{m \times n} \pm [b_{ij}]_{m \times n} = [c_{ij}]_{m \times n},$$

where

$$c_{ij} = a_{ij} \pm b_{ij}.$$

Note:

A matrix minus itself is called a “**null matrix**” and may be denoted, for short, by $[0]_{m \times n}$ or just $[0]$ when the order is understood.

ILLUSTRATION

A grocer has two shops, each selling apples, oranges and bananas.

The sales, in kilogrammes, of each fruit for the two shops on two separate days are represented by the matrices

$$\begin{bmatrix} 36 & 25 & 10 \\ 20 & 30 & 15 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 40 & 30 & 12 \\ 22 & 35 & 20 \end{bmatrix}$$

where the rows refer to the two shops and the columns refer to apples, oranges and bananas respectively.

The total sales, in kilogrammes, of each fruit for the two shops on both days together are represented by the matrix

$$\begin{bmatrix} 76 & 55 & 22 \\ 42 & 65 & 35 \end{bmatrix}.$$

The differences in sales of each fruit for the two shops between the second day and the first day are represented by the matrix

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & 5 & 5 \end{bmatrix}.$$

3. Additive Identities and Additive Inverses

(a) A null matrix, added to another matrix of the same order, leaves it unchanged.

We say that a null matrix behaves as an **“additive identity”**.

(b) Since $A - A = [0]$, we say that $-A$ is the **“additive inverse”** of A .