"JUST THE MATHS"

SLIDES NUMBER

8.6

VECTORS 6
(Vector equations of planes)

by

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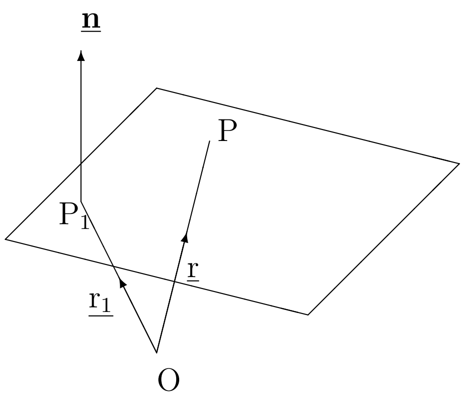
UNIT 8.6 - VECTORS 6

VECTOR EQUATIONS OF PLANES

8.6.1 THE PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN VECTOR

A plane in space is completely specified if we know one point in it, together with a vector which is perpendicular to the plane.



In the diagram, the given point is P_1, with position vector r_1, and the given vector is $\underline{n}$.

The vector, $\underline{P_1P}$, is perpendicular to $\underline{n}$.

Hence,

$$(\underline{r} - \underline{r_1}) \bullet \underline{n} = 0$$

or

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = \underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}} = d \text{ say.}$$

Notes:

(i) When $\underline{\mathrm{n}}$ is a unit vector, d is the perpendicular projection of $\underline{\mathrm{r}_1}$ onto $\underline{\mathrm{n}}$.

That is, d is the perpendicular distance of the origin from the plane.

(ii) If

$$\underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ and } \underline{\mathrm{n}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

the cartesian form for the equation of the above plane will be

$$ax + by + cz = d.$$

EXAMPLE

Determine the vector equation and hence the cartesian equation of the plane, passing through the point with position vector, $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, and perpendicular to the vector $\mathbf{i} - 4\mathbf{j} - \mathbf{k}$.

Solution

The vector equation is

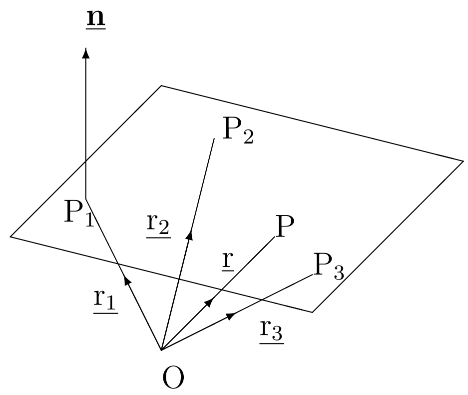
$$\underline{\mathrm{r}} \bullet (\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = (3)(1) + (-2)(-4) + (1)(-1) = 10$$

and, hence, the cartesian equation is

$$x - 4y - z = 10.$$

8.6.2 THE PLANE PASSING THROUGH THREE GIVEN POINTS

We consider a plane passing through the points, $\mathrm{P}_1(x_1, y_1, z_1)$, $\mathrm{P}_2(x_2, y_2, z_2)$ and $\mathrm{P}_3(x_3, y_3, z_3)$.



In the diagram, a suitable vector for $\underline{\mathrm{n}}$ is

$$\underline{\mathrm{P}_1\mathrm{P}_2} \ \mathbf{x} \ \underline{\mathrm{P}_1\mathrm{P}_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}.$$

The equation of the plane is

$$(\underline{r} - \underline{r_1}) \bullet \underline{n} = 0.$$

That is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

From the properties of determinants, this is equivalent to

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

EXAMPLE

Determine the cartesian equation of the plane passing through the three points, $(0, 2, -1)$, $(3, 0, 1)$ and $(-3, -2, 0)$.

Solution

The equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 2 & -1 & 1 \\ 3 & 0 & 1 & 1 \\ -3 & -2 & 0 & 1 \end{vmatrix} = 0,$$

which simplifies to

$$2x - 3y - 6z = 0.$$

This plane also passes through the origin.

8.6.3 THE POINT OF INTERSECTION OF A STRAIGHT LINE AND A PLANE

The vector equation of a straight line passing through the fixed point with position vector, $\underline{\mathrm{r}}_1$, and parallel to the fixed vector, $\underline{\mathrm{a}}$, is

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}}.$$

We require the point at which this line meets the plane

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d.$$

We require t to be such that

$$(\underline{\mathrm{r}_1} + t\underline{\mathrm{a}}) \bullet \underline{\mathrm{n}} = d.$$

From this equation, the value of t and, hence, the point of intersection, may be found.

EXAMPLE

Determine the point of intersection of the plane

$$\underline{\mathrm{r}} \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7$$

and the straight line passing through the point, $(4, -1, 3)$, which is parallel to the vector, $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

Solution

We need to obtain t such that

$$(4\mathbf{i} - \mathbf{j} + 3\mathbf{k} + t[2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}]) \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7.$$

That is,

$$(4+2t)(1)+(-1-2t)(-3)+(3+5t)(-1) = 7 \text{ or } 4+3t = 7.$$

Thus, $t = 1$ and, hence, the point of intersection is $(4 + 2, -1 - 2, 3 + 5) = (6, -3, 8)$.

8.6.4 THE LINE OF INTERSECTION OF TWO PLANES

Let two non-parallel planes have vector equations

$$\underline{r} \bullet \underline{n_1} = d_1 \text{ and } \underline{r} \bullet \underline{n_2} = d_2.$$

Their line of intersection will be perpendicular to both $\underline{n_1}$ and $\underline{n_2}$.

The line of intersection will thus be parallel to $\underline{n_1}$ $\mathbf{x}$ $\underline{n_2}$.

To obtain the vector equation of this line, we must determine a point on it.

For convenience, we take the point (common to both planes) for which one of x, y or z is zero.

EXAMPLE

Determine the vector equation and, hence, the cartesian equations (in standard form), of the line of intersection of the planes whose vector equations are

$$\underline{r} \bullet \underline{n_1} = 2 \text{ and } \underline{r} \bullet \underline{n_2} = 17,$$

where

$$\underline{n_1} = \mathbf{i} + \mathbf{j} + \mathbf{k} \text{ and } \underline{n_2} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

Solution

Firstly,

$$\underline{\mathrm{n}_1} \ \mathbf{x} \ \underline{\mathrm{n}_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}.$$

Secondly, the cartesian equations of the two planes are

$$x + y + z = 2 \ \text{ and } \ 4x + y + 2z = 17.$$

When $z = 0$, these become

$$x + y = 2 \ \text{ and } \ 4x + y = 17.$$

These have common solution $x = 5$, $y = -3$.

Thus, a point on the line of intersection is $(5, -3, 0)$, which has position vector $5\mathbf{i} - 3\mathbf{j}$.

Hence, the vector equation of the line of intersection is

$$\underline{\mathrm{r}} = 5\mathbf{i} - 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}).$$

Finally, since $x = 5 + t$, $y = -3 + 2t$ and $z = -3t$, the line of intersection is represented, in standard cartesian form, by

$$\frac{x-5}{1} = \frac{y+3}{2} = \frac{z}{-3}.$$

8.6.5 THE PERPENDICULAR DISTANCE OF A POINT FROM A PLANE

We consider the plane whose equation is

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d$$

and the point, P_1, whose position vector is $\underline{\mathrm{r}_1}$.

The straight line through the point P_1 which is perpendicular to the plane has equation

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{n}}.$$

This line meets the plane at the point, P_0, with position vector $\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}$, where

$$(\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) \bullet \underline{\mathrm{n}} = d.$$

That is,

$$(\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}}) + t_0\mathrm{n}^2 = d.$$

Hence,

$$t_0 = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}^2}.$$

Finally,

$$\underline{\mathrm{P}_0\mathrm{P}_1} = (\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) - \underline{\mathrm{r}_1} = t_0\underline{\mathrm{n}},$$

and its magnitude, $t_0\mathrm{n}$, will be the perpendicular distance, p, of the point P_1 from the plane.

In other words,

$$p = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}}.$$

Note:
In terms of cartesian co-ordinates, this formula is equivalent to

$$p = \frac{d - (ax_1 + by_1 + cz_1)}{\sqrt{a^2 + b^2 + c^2}}.$$

EXAMPLE

Determine the perpendicular distance, p, of the point $(2, -3, 4)$ from the plane whose cartesian equation is $x + 2y + 2z = 13$.

Solution

From the cartesian formula,

$$p = \frac{13 - [(1)(2) + (2)(-3) + (2)(4)]}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{9}{3} = 3.$$