"JUST THE MATHS"

SLIDES NUMBER

8.4

VECTORS 4
(Triple products)

by

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UNIT 8.4 - VECTORS 4

TRIPLE PRODUCTS

8.4.1 THE TRIPLE SCALAR PRODUCT

DEFINITION 1

Given three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, expressions such as

$$\underline{a} \bullet (\underline{b}\ \mathbf{x}\ \underline{c}),\quad \underline{b} \bullet (\underline{c}\ \mathbf{x}\ \underline{a}),\quad \underline{c} \bullet (\underline{a}\ \mathbf{x}\ \underline{b})$$

or

$$(\underline{a}\ \mathbf{x}\ \underline{b}) \bullet \underline{c},\quad (\underline{b}\ \mathbf{x}\ \underline{c}) \bullet \underline{a},\quad (\underline{c}\ \mathbf{x}\ \underline{a}) \bullet \underline{b}$$

are called **"triple scalar products"** because their results are all scalar quantities.

The brackets are optional because there is no ambiguity without them.

We shall take $\underline{a} \bullet (\underline{b}\ \mathbf{x}\ \underline{c})$ as the typical triple scalar product.

The formula for a triple scalar product

Suppose that

$$\underline{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \ \underline{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},\ \ \underline{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

Then,

$$\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From the basic formula for scalar product,
this becomes

$$\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Notes:

(i) By properties of determinants (interchanging rows),

$$\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}}) = -\underline{\mathrm{a}} \bullet (\underline{\mathrm{c}} \mathbf{\ x\ } \underline{\mathrm{b}}) = \underline{\mathrm{c}} \bullet (\underline{\mathrm{a}} \mathbf{\ x\ } \underline{\mathrm{b}})$$

$$= -\underline{\mathrm{c}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{a}}) = \underline{\mathrm{b}} \bullet (\underline{\mathrm{c}} \mathbf{\ x\ } \underline{\mathrm{a}}) = -\underline{\mathrm{b}} \bullet (\underline{\mathrm{a}} \mathbf{\ x\ } \underline{\mathrm{c}}).$$

The **"cyclic permutations"** of $\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}})$ are all equal in numerical value and in sign; the remaining permutations are equal to $\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}})$ in numerical value, but opposite in sign.

(ii) $\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}})$, is often denoted by $[\underline{\mathrm{a}}, \underline{\mathrm{b}}, \underline{\mathrm{c}}]$.

EXAMPLE

Evaluate the triple scalar product, $\underline{a} \bullet (\underline{b}\ \mathbf{x}\ \underline{c})$, in the case when

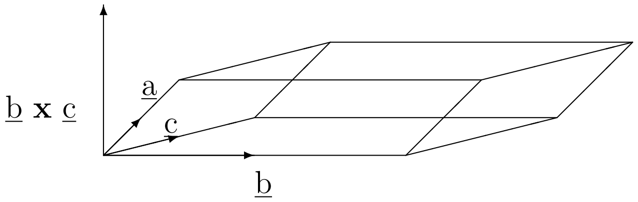
$$\underline{a} = 2\mathbf{i} + \mathbf{k}, \quad \underline{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \ \text{ and } \ \underline{c} = -\mathbf{i} + \mathbf{j}.$$

Solution

$$\underline{a} \bullet (\underline{b}\ \mathbf{x}\ \underline{c}) = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 2.(-2) - 0.(2) + 1.(2) = -2.$$

A geometrical application of the triple scalar product

Suppose that $\underline{a}$, $\underline{b}$ and $\underline{c}$ lie along three adjacent edges of a parallelepiped.



The area of the base of the parallelepiped is the **magnitude** of the vector $\underline{b}\ \mathbf{x}\ \underline{c}$ which is perpendicular to the base.

The perpendicular height of the parallelepiped is the projection of the vector $\underline{a}$ onto the vector $\underline{b}\ \mathbf{x}\ \underline{c}$.

The perpendicular height is

$$\frac{\underline{a} \bullet (\underline{b} \mathbf{x}\, \underline{c})}{|\underline{b} \mathbf{x}\, \underline{c}|}.$$

The volume, V, of the parallelepiped is equal to the area of the base times the perpendicular height.

Hence,

$$V = \underline{a} \bullet (\underline{b} \mathbf{x}\, \underline{c}).$$

This is the result **numerically**, since the triple scalar product could turn out to be negative.

Note:
The above geometrical application also provides a condition that three given vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$ lie in the same plane.

The condition that they are **"coplanar"** is that

$$\underline{a} \bullet (\underline{b} \mathbf{x}\, \underline{c}) = 0.$$

That is, the three vectors would determine a parallelepiped whose volume is zero.

8.4.2 THE TRIPLE VECTOR PRODUCT

DEFINITION 2

If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three vectors, then the expression

$$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c})$$

is called the **"triple vector product"** of $\underline{a}$ with $\underline{b}$ and $\underline{c}$.

Notes:

(i) The triple vector product is clearly a vector quantity.

(ii) The brackets are important since it can be shown (in general) that

$$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c}) \neq (\underline{a} \textbf{ x } \underline{b}) \textbf{ x } \underline{c}.$$

ILLUSTRATION

Let the three vectors be position vectors, with the origin as a common end-point.

Then,

$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c})$ is perpendicular to both $\underline{a}$ and $\underline{b} \textbf{ x } \underline{c}$.

But $\underline{b} \textbf{ x } \underline{c}$ is already perpendicular to both $\underline{b}$ and $\underline{c}$.

That is, $\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c})$ lies in the plane of $\underline{b}$ and $\underline{c}$.

Consequently, $(\underline{a}\ \mathbf{x}\ \underline{b})\ \mathbf{x}\ \underline{c}$, which is the same as $-\underline{c}\ \mathbf{x}\ (\underline{a}\ \mathbf{x}\ \underline{b})$, will lie in the plane of $\underline{a}$ and $\underline{b}$.

Hence, $(\underline{a}\ \mathbf{x}\ \underline{b})\ \mathbf{x}\ \underline{c}$ will, in general, be different from $\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c})$.

The formula for a triple vector product

Suppose that

$$\underline{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \ \underline{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},\ \ \underline{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

Then,

$$\underline{a}\ \mathbf{x}\ (\underline{b}\ \mathbf{x}\ \underline{c}) = \underline{a}\ \mathbf{x} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix}.$$

The $\mathbf{i}$ component of this vector is equal to

$$a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) =$$
$$b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3).$$

By adding and subtracting $a_1b_1c_1$, the expression $b_1(a_2c_2+a_3c_3) - c_1(a_2b_2 + a_3b_3)$ can be rearranged in the form

$$(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1.$$

This is the **i** component of the vector

$$(\underline{\mathrm{a}} \bullet \underline{\mathrm{c}})\underline{\mathrm{b}} - (\underline{\mathrm{a}} \bullet \underline{\mathrm{b}})\underline{\mathrm{c}}.$$

Similar expressions can be obtained for the **j** and **k** components.

We conclude that

$$\underline{\mathrm{a}}\ \mathbf{x}\ (\underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{c}}) = (\underline{\mathrm{a}} \bullet \underline{\mathrm{c}})\underline{\mathrm{b}} - (\underline{\mathrm{a}} \bullet \underline{\mathrm{b}})\underline{\mathrm{c}}.$$

EXAMPLE

Determine the triple vector product of $\underline{\mathrm{a}}$ with $\underline{\mathrm{b}}$ and $\underline{\mathrm{c}}$, where

$$\underline{\mathrm{a}} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \underline{\mathrm{b}} = -2\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad \underline{\mathrm{c}} = 3\mathbf{k}.$$

Solution

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{c}} = -3 \quad \text{and} \quad \underline{a} \bullet \underline{\mathrm{b}} = 4.$$

$$\text{Hence,} \quad \underline{\mathrm{a}}\ \mathbf{x}\ (\underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{c}}) = -3\underline{\mathrm{b}} - 4\underline{\mathrm{c}} = 6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k}.$$