"JUST THE MATHS"

SLIDES NUMBER

8.1

VECTORS 1
(Introduction to vector algebra)

by

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UNIT 8.1 - VECTORS 1 - INTRODUCTION TO VECTOR ALGEBRA

8.1.1 DEFINITIONS

1. A **“scalar”** quantity is one which has magnitude, but is not related to any direction in space.

   Examples: Mass, Speed, Area, Work.

2. A **“vector”** quantity is one which is specified by both a magnitude and a direction in space.

   Examples: Velocity, Weight, Acceleration.

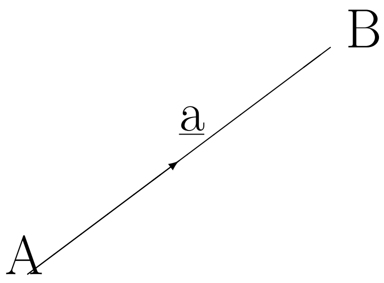
3. A vector quantity with a fixed point of application is called a **“position vector”**.

4. A vector quantity which is restricted to a fixed line of action is called a **“line vector”**.

5. A vector quantity which is defined only by its magnitude and direction is called a **“free vector”**.

   Note:

   Unless otherwise stated, all vectors in the remainder of these units will be free vectors.



6. A vector quantity can be represented diagramatically by a directed straight line segment in space (with an arrow head) whose direction is that of the vector and whose length represents is magnitude according to a suitable scale.

7. The symbols $\underline{a}$, $\underline{b}$, $\underline{c}$, ... will be used to denote vectors with magnitudes a,b,c....

   Sometimes we use $\underline{AB}$ for the vector drawn from the point A to the point B.

   Notes:

   (i) The magnitude of the vector $\underline{AB}$ is the length of the line AB.

   It can also be denoted by the symbol $|\underline{AB}|$.

   (ii) The magnitude of the vector $\underline{a}$ is the number a.

   It can also be denoted by the symbol $|\underline{a}|$.

8. A vector whose magnitude is 1 is called a **"unit vector"**.

   The symbol $\underline{\hat{a}}$ denotes a unit vector in the same direction as $\underline{a}$.

   A vector whose magnitude is zero is called a **"zero vector"** and is denoted by **O** or $\underline{O}$. It has indeterminate direction.

9. Two (free) vectors $\underline{a}$ and $\underline{b}$ are said to be **"equal"** if they have the same magnitude and direction.

   Note:

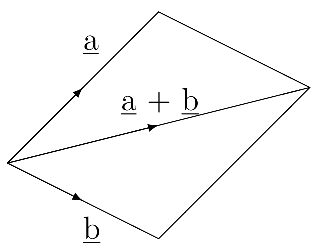
   Two directed straight line segments which are parallel and equal in length represent exactly the same vector.

10. A vector whose magnitude is that of $\underline{a}$ but with opposite direction is denoted by $-\underline{a}$.

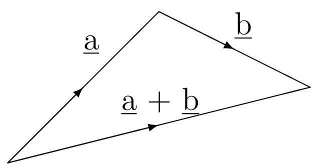
8.1.2 ADDITION AND SUBTRACTION OF VECTORS

We define the sum of two arbitrary vectors diagramatically using either a parallelogram or a triangle.

This will then lead also to a definition of subtraction for two vectors.



Parallelogram Law



Triangle Law

Notes:
(i) The Triangle Law is more widely used than the Parallelogram Law.

$\underline{a}$ and $\underline{b}$ describe the triangle in the same sense.

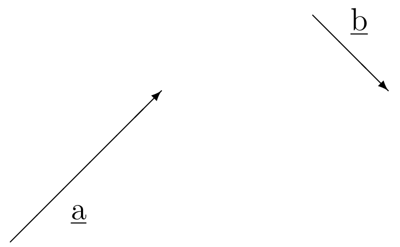
$\underline{a} + \underline{b}$ describes the triangle in the opposite sense.

(ii) To define subtraction, we use

$$\underline{a} - \underline{b} = \underline{a} + (-\underline{b}).$$

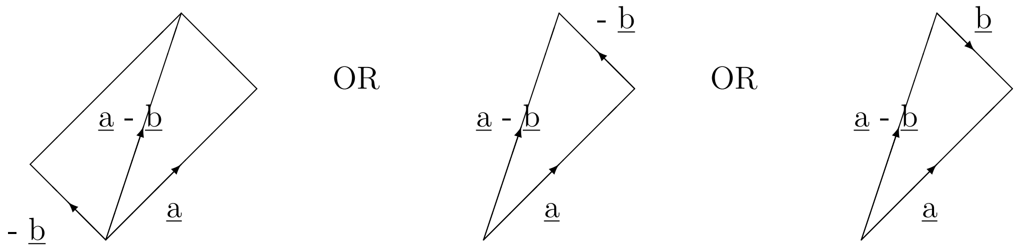
EXAMPLE

Determine $\underline{a} - \underline{b}$ for the following vectors:



Solution

We may construct the following diagrams:



Observations

(i) To determine $\underline{a} - \underline{b}$, we require that $\underline{a}$ and $\underline{b}$ describe the triangle in <u>opposite</u> senses while $\underline{a} - \underline{b}$ describes the triangle in the same sense as $\underline{b}$.

(ii) The sum of the three vectors describing the sides of a triangle in the same sense is the zero vector.

8.1.3 MULTIPLICATION OF A VECTOR BY A SCALAR

If m is any positive real number, $m\underline{\text{a}}$ is defined to be a vector in the same direction as $\underline{\text{a}}$, but of m times its magnitude.

$-m\underline{\text{a}}$ is a vector in the opposite direction to $\underline{\text{a}}$, but of m times its magnitude.

Note:
$\underline{\text{a}} = \text{a}\underline{\hat{\text{a}}}$ and hence

$$\frac{1}{\text{a}}.\underline{\text{a}} = \underline{\hat{\text{a}}}.$$

If any vector is multiplied by the reciprocal of its magnitude, we obtain a unit vector in the same direction.

This process is called **"normalising the vector"**.

8.1.4 LAWS OF ALGEBRA OBEYED BY VECTORS

(i) The Commutative Law of Addition

$$\underline{\mathrm{a}} + \underline{\mathrm{b}} = \underline{\mathrm{b}} + \underline{\mathrm{a}}.$$

(ii) The Associative Law of Addition

$$\underline{\mathrm{a}} + (\underline{\mathrm{b}} + \underline{\mathrm{c}}) = (\underline{\mathrm{a}} + \underline{\mathrm{b}}) + \underline{\mathrm{c}} = \underline{\mathrm{a}} + \underline{\mathrm{b}} + \underline{\mathrm{c}}.$$

(iii) The Associative Law of Multiplication by a Scalar

$$m(n\underline{\mathrm{a}}) = (mn)\underline{\mathrm{a}} = mn\underline{\mathrm{a}}.$$

(iv) The Distributive Laws for Multiplication by a Scalar

$$(m + n)\underline{\mathrm{a}} = m\underline{\mathrm{a}} + n\underline{\mathrm{a}}$$

and

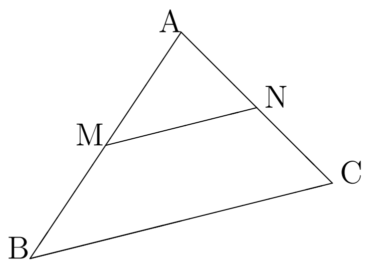
$$m(\underline{\mathrm{a}} + \underline{\mathrm{b}}) = m\underline{\mathrm{a}} + m\underline{\mathrm{b}}.$$

8.1.5 VECTOR PROOFS OF GEOMETRICAL RESULTS

EXAMPLES

1. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of its length.

   Solution



By the Triangle Law,

$$\underline{\mathrm{BC}} = \underline{\mathrm{BA}} + \underline{\mathrm{AC}}$$

and

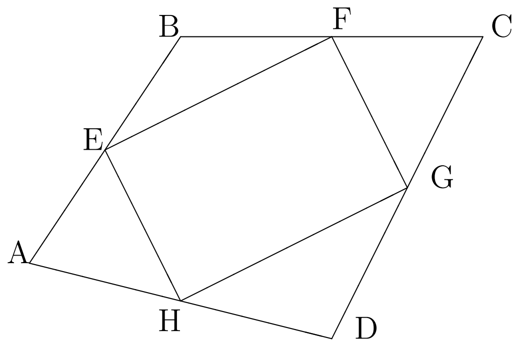
$$\underline{\mathrm{MN}} = \underline{\mathrm{MA}} + \underline{\mathrm{AN}} = \frac{1}{2}\underline{\mathrm{BA}} + \frac{1}{2}\underline{\mathrm{AC}}.$$

Hence,

$$\underline{\mathrm{MN}} = \frac{1}{2}(\underline{\mathrm{BA}} + \underline{\mathrm{AC}}) = \frac{1}{2}\underline{\mathrm{BC}}.$$

2. ABCD is a quadrilateral (four-sided figure) and E,F,G,H are the midpoints of AB, BC, CD and DA respectively. Show that EFGH is a parallelogram.

Solution



By the Triangle Law,

$$\underline{EF} = \underline{EB} + \underline{BF} = \frac{1}{2}\underline{AB} + \frac{1}{2}\underline{BC} = \frac{1}{2}(\underline{AB} + \underline{BC}) = \frac{1}{2}\underline{AC}$$

and also

$$\underline{HG} = \underline{HD} + \underline{DG} = \frac{1}{2}\underline{AD} + \frac{1}{2}\underline{DC} = \frac{1}{2}(\underline{AD} + \underline{DC}) = \frac{1}{2}\underline{AC}.$$

Hence,

$$\underline{EF} = \underline{HG}.$$