"JUST THE MATHS"

SLIDES NUMBER

7.1

DETERMINANTS 1
(Second order determinants)

by

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UNIT 7.1 - DETERMINANTS 1

SECOND ORDER DETERMINANTS

7.1.1 PAIRS OF SIMULTANEOUS LINEAR EQUATIONS

Determinants may be introduced by considering

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0. - - - - - - - - - -(2) \end{aligned}$$

Subtracting equation (2) × b_1 from
equation (1) × b_2,

$$a_1b_2x - a_2b_1x + c_1b_2 - c_2b_1 = 0.$$

$$\text{Hence, } \; x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \; \text{ provided } \; a_1b_2 - a_2b_1 \neq 0.$$

Subtracting equation (2) × a_1 from
equation (1) × a_2,

$$a_2b_1y - a_1b_2y + a_2c_1 - a_1c_2 = 0.$$

$$\text{Hemce, } \; y = -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \; \text{ provided } \; a_1b_2 - a_2b_1 \neq 0.$$

The Symmetrical Form

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1},$$

provided $a_1b_2 - a_2b_1 \neq 0$.

7.1.2 THE DEFINITION OF A SECOND ORDER DETERMINANT

Let

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC.$$

The symbol on the left-hand-side may be called either a **"second order determinant"** or a **"**2 × 2 **determinant"**;

it has two **"rows"** (horizontally), two **"columns"** (vertically) and four **"elements"** (the numbers inside the determinant).

7.1.3 CRAMER'S RULE FOR TWO SIMULTANEOUS LINEAR EQUATIONS

The symmetrical solution to the two simultaneous linear equations may now be written

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

provided $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$;

or, in an abbreviated form,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

provided $\Delta_0 \neq 0$.

This determinant rule for solving two simultaneous linear equations is called **"Cramer's Rule"** and has equivalent forms for a larger number of equations.

Note:
The interpretation of Cramer's Rule in the case when $a_1b_2 - a_2b_1 = 0$ is a special case.

Observations

In Cramer's Rule,

1. To remember the determinant underneath x, cover up the x terms in the original simultaneous equations.
2. To remember the determinant underneath y, cover up the y terms in the original simultaneous equations.
3. To remember the determinant underneath 1 cover up the constant terms in the original simultaneous equations.
4. The final determinant is labelled Δ_0 as a reminder to evaluate it **first**.

   If $\Delta_0 = 0$, there is no point in evaluating Δ_1 and Δ_2.

EXAMPLES

1. Evaluate the determinant

$$\Delta = \begin{vmatrix} 7 & -2 \\ 4 & 5 \end{vmatrix}.$$

Solution

$$\Delta = 7 \times 5 - 4 \times (-2) = 35 + 8 = 43.$$

2. Express the value of the determinant

$$\Delta = \begin{vmatrix} -p & -q \\ p & -q \end{vmatrix}$$

in terms of p and q.

Solution

$$\Delta = (-p) \times (-q) - p \times (-q) = p.q + p.q = 2pq$$

3. Use Cramer's Rule to solve for x and y the simultaneous linear equations

$$\begin{aligned} 5x - 3y &= -3, \\ 2x - y &= -2. \end{aligned}$$

Solution

Rearrange the equations in the form

$$\begin{aligned} 5x - 3y + 3 &= 0, \\ 2x - y + 2 &= 0. \end{aligned}$$

Hence, by Cramer's Rule,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

where

$$\Delta_0 = \begin{vmatrix} 5 & -3 \\ 2 & -1 \end{vmatrix} = -5 + 6 = 1;$$

$$\Delta_1 = \begin{vmatrix} -3 & 3 \\ -1 & 2 \end{vmatrix} = -6 + 3 = -3;$$

$$\Delta_2 = \begin{vmatrix} 5 & 3 \\ 2 & 2 \end{vmatrix} = 10 - 6 = 4.$$

Thus,

$$x = \frac{\Delta_1}{\Delta_0} = -3 \ \text{ and } \ y = -\frac{\Delta_2}{\Delta_0} = -4.$$

Special Cases

If $\Delta_0 = 0$, then the equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, ----------(1) \\ a_2x + b_2y + c_2 &= 0. ----------(2) \end{aligned}$$

are such that

$$a_1b_2 - a_2b_1 = 0.$$

In other words,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

The x and y terms in one of the equations are proportional to the x and y terms in the other equation.

Two situations arise:

EXAMPLES

1. For the set of equations

$$\begin{aligned} 3x - 2y &= 5, \\ 6x - 4y &= 10, \end{aligned}$$

$\Delta_0 = 0$ but the second equation is simply a multiple of the first.

One of the equations is redundant and so there exists an **infinite number of solutions**.

Either of the variables may be chosen at random with the remaining variable being expressible in terms of it.

2. For the set of equations

$$\begin{aligned} 3x - 2y &= 5, \\ 6x - 4y &= 7, \end{aligned}$$

$\Delta_0 = 0$ but, from the second equation,

$$3x - 2y = 3.5,$$

which is inconsistent with

$$3x - 2y = 5.$$

In this case **there are no solutions at all**.

Summary of the Special Cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.