"JUST THE MATHS"

SLIDES NUMBER

6.6

COMPLEX NUMBERS 6
(Complex loci)

by

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UNIT 6.6 - COMPLEX NUMBERS 6

COMPLEX LOCI

6.6.1 INTRODUCTION

The directed line segment joining the point representing a complex number z_1 to the point representing a complex number z_2 is of length equal to $|z_2 - z_1|$ and is inclined to the positive direction of the real axis at an angle equal to $\arg(z_2 - z_1)$. (See Unit 6.2).

Variable complex numbers may be constrained to move along a certain path (or **"locus"**) in the Argand Diagram.

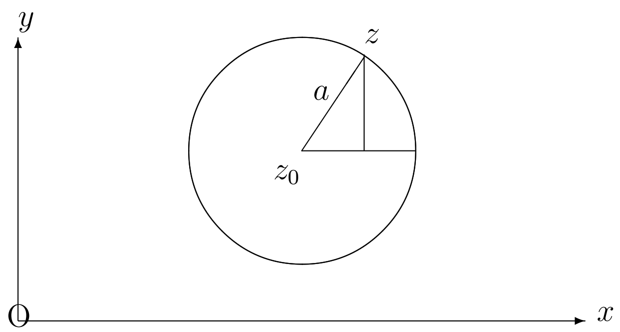
For many practical applications, such paths (or **"loci"**) will normally be either straight lines or circles.

Let $z = x + jy$ denote a **variable** complex number (represented by the point (x, y) in the Argand Diagram).

Let $z_0 = x_0 + jy_0$ denote a **fixed** complex number (represented by the point (x_0, y_0) in the Argand Diagram).

6.6.2 THE CIRCLE

Let the moving point representing z move on a circle, with radius a, whose centre is at the fixed point representing z_0.



Then,

$$|z - z_0| = a.$$

Note:
Substituting $z = x + jy$ and $z_0 = x_0 + jy_0$,

$$|(x - x_0) + j(y - y_0)| = a,$$

$$\text{or} \quad (x - x_0)^2 + (y - y_0)^2 = a^2.$$

ILLUSTRATION

$$|z - 3 + j4| = 7$$

represents a circle, with radius 7, whose centre is the point representing the complex number $3 - j4$.

In cartesian co-ordinates,

$$(x - 3)^2 + (y + 4)^2 = 49.$$

6.6.3 THE HALF-STRAIGHT-LINE

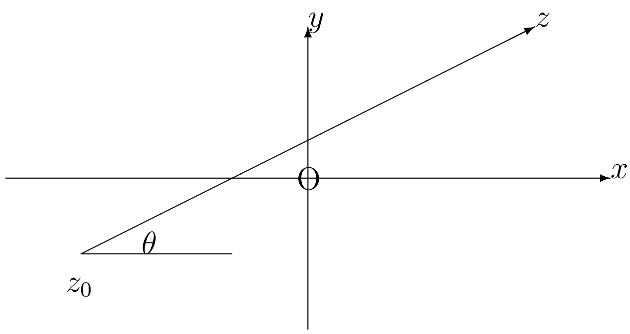
Let the **"directed"** straight line segment described **from** the fixed point representing z_0 **to** the moving point representing z be inclined at an angle θ to the positive direction of the real axis.

Then,

$$\arg(z - z_0) = \theta.$$

This equation is satisfied by **all** of the values of z for which the inclination of the directed line segment is genuinely θ and **not** $180^\circ - \theta$

$180^\circ - \theta$ corresponds to points on the other half of the straight line joining the two points.



Note:
Substituting $z = x + jy$ and $z_0 = x_0 + jy_0$,

$$\arg([x - x_0] + j[y - y_0]) = \theta.$$

That is,

$$\tan^{-1}\frac{y - y_0}{x - x_0} = \theta,$$

or

$$y - y_0 = \tan\theta(x - x_0).$$

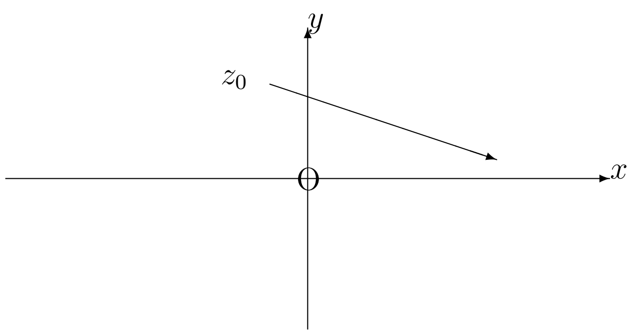
This is the equation of a straight line with gradient $\tan\theta$ passing through the point (x_0, y_0).

It represents only that half of the straight line for which $x - x_0$ and $y - y_0$ correspond, in sign as well as value, to the real and imaginary parts of a complex number whose argument is genuinely θ and not $180^\circ - \theta$.

ILLUSTRATION

$$\arg(z + 1 - j5) = -\frac{\pi}{6}$$

represents the half-straight-line described from the point representing $z_0 = -1 + j5$ to the point representing $z = x + jy$ and inclined to the positive direction of the real axis at an angle of $-\frac{\pi}{6}$.



In terms of cartesian co-ordinates,

$$\arg([x + 1] + j[y - 5]) = -\frac{\pi}{6}.$$

We need $x + 1 > 0$ and $y - 5 < 0$.

We thus have the half-straight-line with equation

$$y - 5 = \tan\left(-\frac{\pi}{6}\right)(x + 1) = -\frac{1}{\sqrt{3}}(x + 1),$$

which lies to the right of, and below the point $(-1, 5)$.

6.6.4 MORE GENERAL LOCI

In general, we substitute $z = x + jy$ to obtain the cartesian equation of the locus.

ILLUSTRATIONS

1. The equation

$$\left|\frac{z-1}{z+2}\right| = 3$$

may be written

$$|z-1| = 3|z+2|.$$

That is,

$$(x-1)^2 + y^2 = 3[(x+2)^2 + y^2],$$

which simplifies to

$$2x^2 + 2y^2 + 14x + 13 = 0,$$

or

$$\left(x + \frac{7}{2}\right)^2 + y^2 = \frac{23}{4},$$

representing a circle with centre $\left(-\frac{7}{2}, 0\right)$ and radius $\sqrt{\frac{23}{4}}$.

2. The equation

$$\arg\left(\frac{z-3}{z}\right) = \frac{\pi}{4}$$

may be written

$$\arg(z-3) - \arg z = \frac{\pi}{4}.$$

That is,

$$\arg([x-3]+jy) - \arg(x+jy) = \frac{\pi}{4}.$$

Taking tangents of both sides and using the trigonometric identity for $\tan(A-B)$,

$$\frac{\frac{y}{x-3} - \frac{y}{x}}{1 + \frac{y}{x-3}\frac{y}{x}} = 1.$$

On simplification,

$$x^2 + y^2 - 3x - 3y = 0,$$

or

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{2},$$

the equation of a circle with centre $\left(\frac{3}{2}, \frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$.

But $\frac{z-3}{z}$ cannot have an argument of $\frac{\pi}{4}$ unless its real and imaginary parts are **both** positive.

In fact,

$$\frac{z-3}{z} = \frac{(x-3)+jy}{x+jy}.\frac{x-jy}{x-jy} = \frac{x(x-3)+y^2+j3}{x^2+y^2},$$

which requires

$$x(x-3)+y^2 > 0.$$

That is,

$$x^2+y^2-3x > 0,$$

or

$$\left(x-\frac{3}{2}\right)^2 + y^2 > \frac{9}{4}.$$

Conclusion

The locus is that part of the circle with centre $\left(\frac{3}{2},\frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$ which lies **outside** the circle with centre $\left(\frac{3}{2},0\right)$ and radius $\frac{3}{2}$.