"JUST THE MATHS"

SLIDES NUMBER

6.5

COMPLEX NUMBERS 5
(Applications to trigonometric identities)

by

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UNIT 6.5 - COMPLEX NUMBERS 5

APPLICATIONS TO TRIGONOMETRIC IDENTITIES

6.5.1 INTRODUCTION

Reminder: **"Pascal's Triangle"**.

If n is a positive whole number, the diagram

$$
\begin{array}{c}
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1
\end{array}
$$

provides the coefficients in the expansion of $(A+B)^n$.

The sequence of terms is

$$A^n,\ A^{n-1}B,\ A^{n-2}B^2,\ A^{n-3}B^3, ..., B^n.$$

EXAMPLE

$$(A+B)^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3.$$

6.5.2 EXPRESSIONS FOR $\cos n\theta$ AND $\sin n\theta$ IN TERMS OF $\cos\theta$ AND $\sin\theta$.

From De Moivre's Theorem,

$$(\cos\theta + j\sin\theta)^n \equiv \cos n\theta + j\sin n\theta.$$

Real part of L.H.S. $\equiv \cos n\theta$.

Imaginary part of L.H.S $\equiv \sin n\theta$.

EXAMPLE

$$(\cos\theta + j\sin\theta)^3 \equiv$$

$$\cos^3\theta + 3\cos^2\theta.(j\sin\theta) + 3\cos\theta.(j\sin\theta)^2 + (j\sin\theta)^3.$$

Hence,

$$\cos 3\theta \equiv \cos^3\theta - 3\cos\theta.\sin^2\theta$$

or $4\cos^3\theta - 3\cos\theta$, using $\sin^2\theta \equiv 1 - \cos^2\theta$;

$$\sin 3\theta \equiv 3\cos^2\theta.\sin\theta - \sin^3\theta$$

or $3\sin\theta - 4\sin^3\theta$, using $\cos^2\theta \equiv 1 - \sin^2\theta$.

6.5.3 EXPRESSIONS FOR $\cos^n\theta$ AND $\sin^n\theta$ IN TERMS OF SINES AND COSINES OF WHOLE MULTIPLES OF θ.

Particularly useful in calculus problems.

Suppose

$$z \equiv \cos\theta + j\sin\theta \quad - \quad (1)$$

Then, by De Moivre's Theorem, or by direct manipulation,

$$\frac{1}{z} \equiv \cos\theta - j\sin\theta \quad - \quad (2)$$

Adding (1) and (2) together, then subtracting (2) from (1), we obtain

$z + \frac{1}{z} \equiv 2\cos\theta$	$z - \frac{1}{z} \equiv j2\sin\theta$

Also, by De Moivre's Theorem,

$$z^n \equiv \cos n\theta + j\sin n\theta \quad - \quad (3)$$

and

$$\frac{1}{z^n} \equiv \cos n\theta - j\sin n\theta \quad - \quad (4)$$

Adding (3) and (4) together, then subtracting (4) from (3), we obtain

$z^n + \frac{1}{z^n} \equiv 2\cos n\theta$	$z^n - \frac{1}{z^n} \equiv j2\sin n\theta$

Note:

This last result <u>includes</u> the previous one for $n = 1$.

EXAMPLES

1. Determine an identity for $\sin^3\theta$.

Solution

$$j^3 2^3 \sin^3\theta \equiv \left(z - \frac{1}{z}\right)^3,$$

where $z \equiv \cos\theta + j\sin\theta$.

That is,

$$-j8\sin^3\theta \equiv z^3 - 3z^2.\frac{1}{z} + 3z.\left(\frac{1}{z}\right)^2 - \frac{1}{z^3}$$

or, after cancelling common factors,

$$-j8\sin^3\theta \equiv z^3 - 3z + \frac{3}{z} - \frac{1}{z^3} \equiv \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right),$$

which gives

$$-j8\sin^3\theta \equiv j2\sin 3\theta - j6\sin\theta.$$

Hence,

$$\sin^3\theta \equiv \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right).$$

2. Determine an identity for $\cos^4\theta$.

Solution

$$2^4\cos^4\theta \equiv \left(z + \frac{1}{z}\right)^4,$$

where $z \equiv \cos\theta + j\sin\theta$.

That is,

$$16\cos^4\theta \equiv z^4 + 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 + 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

or, after cancelling common factors,

$$16\cos^4\theta \equiv z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4} \equiv z^4+\frac{1}{z^4}+4\left(z^2+\frac{1}{z^2}\right)+6,$$

which gives

$$16\cos^4\theta \equiv 2\cos 4\theta + 8\cos 2\theta + 6.$$

Hence,

$$\cos^4\theta \equiv \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right).$$