"JUST THE MATHS"

SLIDES NUMBER

6.3

COMPLEX NUMBERS 3
(The polar & exponential forms)

by

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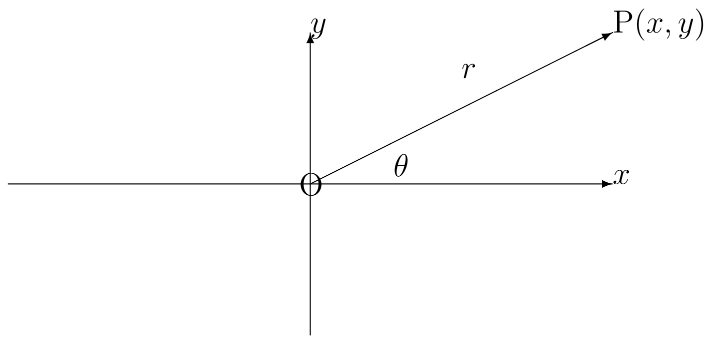
UNIT 6.3 - COMPLEX NUMBERS 3

THE POLAR AND EXPONENTIAL FORMS

6.3.1 THE POLAR FORM



From the diagram,

$$\frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{y}{r} = \sin\theta;$$

$$x = r\cos\theta, \quad y = r\sin\theta;$$

$$x + jy = r(\cos\theta + j\sin\theta).$$

$x + jy$ is called the **"rectangular form"** or **"cartesian form"**.

$r(\cos\theta + j\sin\theta)$ ($r\angle\theta$ for short) is called the **"polar form"**.

θ may be positive, negative or zero and may be expressed in either degrees or radians.

EXAMPLES

1. Express the complex number $z = \sqrt{3} + j$ in polar form.

   Solution

   $$|z| = r = \sqrt{3+1} = 2$$

   and

   $$\text{Arg}z = \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ + k360^\circ,$$

   where k may be any integer.

   Alternatively, using radians,

   $$\text{Arg}z = \frac{\pi}{6} + k2\pi,$$

   where k may be any integer.

   Hence, in polar form, $z =$

   $2(\cos[30^\circ + k360^\circ] + j\sin[30^\circ + k360^\circ]) = 2\angle[30^\circ + k360^\circ]$

Alternatively, $z =$

$2\left(\cos\left[\frac{\pi}{6} + k2\pi\right] + j\sin\left[\frac{\pi}{6} + k2\pi\right]\right) = 2\angle\left[\frac{\pi}{6} + k2\pi\right].$

2. Express the complex number $z = -1 - j$ in polar form.

Solution

$$|z| = r = \sqrt{1+1} = \sqrt{2}$$

and

$$\text{Arg}z = \theta = \tan^{-1}(1) = -135^\circ + k360^\circ,$$

where k may be any integer.

Alternatively

$$\text{Arg}z = -\frac{3\pi}{4} + k2\pi,$$

where k may be any integer.

Hence, in polar form,

$$z = \sqrt{2}(\cos[-135^\circ + k360^\circ] + j\sin[-135^\circ + k360^\circ])$$

$$= \sqrt{2}\angle[-135^\circ + k360^\circ]$$

or

$$z = \sqrt{2}\left(\cos\left[-\frac{3\pi}{4} + k2\pi\right] + j\sin\left[-\frac{3\pi}{4} + k2\pi\right]\right)$$

$$= \sqrt{2}\angle\left[-\frac{3\pi}{4} + k2\pi\right].$$

Note:
If it is required that the polar form should contain only the **principal** value of the argument, θ, then, provided $-180^\circ < \theta \leq 180^\circ$ or $-\pi < \theta \leq \pi$, the component $k360^\circ$ or $k2\pi$ of the result is simply omitted.

6.3.2 THE EXPONENTIAL FORM

It may be shown that

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$$

These are the **definitions** of $e^z, \sin z$ and $\cos z$.

In the equivalent series for $\sin x$ and $\cos x$, the value x (real), must be in **radians and not degrees**.

Deductions

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + ...$$

But $j^2 = -1$, so

$$e^{j\theta} = 1 + j\frac{\theta}{1!} - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ...$$

Hence,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

provided θ is expressed in radians and not degrees.

The complex number $x + jy$, having modulus r and argument $\theta + k2\pi$ may thus be expressed not only in polar form but also in

the exponential form, $re^{j\theta}$.

ILLUSTRATIONS

1.

$$\sqrt{3}+j=2e^{j\left(\frac{\pi}{6}+k2\pi\right)}.$$

2.

$$-1+j=\sqrt{2}e^{j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

3.

$$-1-j=\sqrt{2}e^{-j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

Note:

If it is required that the exponential form should contain only the **principal** value of the argument, θ, then, provided $-\pi<\theta\leq\pi$, the component $k2\pi$ of the result is simply omitted.

6.3.3 PRODUCTS AND QUOTIENTS IN POLAR FORM

Let

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1) = r_1\angle\theta_1$$

and

$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2) = r_2\angle\theta_2.$$

(a) The Product

$$z_1.z_2 = r_1.r_2(\cos\theta_1 + j\sin\theta_1).(\cos\theta_2 + j\sin\theta_2)$$

That is,

$$z_1.z_2 = r_1.r_2([\cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2]$$

$$+j[\sin\theta_1.\cos\theta_2 + \cos\theta_1.\sin\theta_2]).$$

$$z_1.z_2 = r_1.r_2(\cos[\theta_1+\theta_2]+j\sin[\theta_1+\theta_2]) = r_1.r_2\angle[\theta_1+\theta_2].$$

To determine the product of two complex numbers in polar form, we construct the product of their modulus values and the sum of their argument values.

(b) The Quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + j\sin\theta_1)}{(\cos\theta_2 + j\sin\theta_2)}.$$

Multiplying the numerator and denominator by $\cos\theta_2 - j\sin\theta_2$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}([\cos\theta_1.\cos\theta_2 + \sin\theta_1.\sin\theta_2]$$

$$+j[\sin\theta_1.\cos\theta_2 - \cos\theta_1.\sin\theta_2]).$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos[\theta_1 - \theta_2] + j\sin[\theta_1 - \theta_2]) = \frac{r_1}{r_2}\angle[\theta_1 - \theta_2].$$

To determine the quotient of two complex numbers in polar form, we construct the quotient of their modulus values and the difference of their argument values.

ILLUSTRATIONS

1.

$$(\sqrt{3}+j).(-1-j) = 2\angle 30^\circ.\sqrt{2}\angle(-135^\circ)$$

$$= 2\sqrt{2}\angle(-105^\circ).$$

For all of the complex numbers in this example, including the result, the argument appears as the principal value.

2.

$$\frac{\sqrt{3}+j}{-1-j} = \frac{2\angle 30^\circ}{\sqrt{2}\angle(-135^\circ)} = \sqrt{2}\angle 165^\circ.$$

For all of the complex numbers in this example, including the result, the argument appears as the principal value.

3.

$$(-1-j).(-\sqrt{3}-j) = \sqrt{2}\angle(-135^\circ).2\angle(-150^\circ)$$

$$= 2\sqrt{2}\angle(-285^\circ).$$

This must be converted to $2\sqrt{2}\angle(75^\circ)$ if the principal value of the argument is required.