"JUST THE MATHS"

SLIDES NUMBER

5.9

GEOMETRY 9
(Curve sketching in general)

by

A.J.Hobson

5.9.1 Symmetry
5.9.2 Intersections with the co-ordinate axes
5.9.3 Restrictions on the range of either variable
5.9.4 The form of the curve near the origin
5.9.5 Asymptotes

UNIT 5.9 - GEOMETRY 8

CURVE SKETCHING IN GENERAL

Introduction

Here, we consider the approximate shape of a curve, whose equation is known, rather than an accurate "plot".

5.9.1 SYMMETRY

A curve is symmetrical about the x-axis if its equation contains only even powers of y.

A curve is symmetrical about the y-axis if its equation contains only even powers of x.

A curve is symmetrical with respect to the origin if its equation is unaltered when both x and y are changed in sign.

Symmetry with respect to the origin means that, if a point (x, y) lies on the curve, so does the point $(-x, -y)$.

ILLUSTRATIONS

1. The curve
$$x^2\left(y^2-2\right)=x^4+4$$
is symmetrical about both the x-axis and the y-axis.

   Once the shape of the curve is known in the first quadrant, the rest of the curve is obtained from this part by reflecting it in both axes.

   The curve is also symmetrical with respect to the origin.

2. The curve
$$xy=5$$
is symmetrical with respect to the origin but not about either of the co-ordinate axes.

5.9.2 INTERSECTIONS WITH THE CO-ORDINATE AXES

Any curve intersects the x-axis where $y=0$ and the y-axis where $x=0$.

Sometimes the curve has no intersection with one or more of the co-ordinate axes.

This will be borne out by an inability to solve for x when $y=0$ or y when $x=0$ (or both).

ILLUSTRATION

The circle,

$$x^2 + y^2 - 4x - 2y + 4 = 0,$$

meets the x-axis where

$$x^2 - 4x + 4 = 0.$$

That is,

$$(x - 2)^2 = 0,$$

giving a double intersection at the point $(2, 0)$.

This means that the circle **touches** the x-axis at $(2, 0)$.

The circle meets the y-axis where

$$y^2 - 2y + 4 = 0.$$

That is,

$$(y - 1)^2 = -3,$$

which is impossible, since the left hand side is bound to be positive when y is a real number.

Thus there are no intersections with the y-axis.

5.9.3 RESTRICTIONS ON THE RANGE OF EITHER VARIABLE

We illustrate as follows:

ILLUSTRATIONS

1. The curve whose equation is

$$y^2 = 4x$$

requires that x shall not be negative;
that is, $x \geq 0$.

2. The curve whose equation is

$$y^2 = x\left(x^2 - 1\right)$$

requires that the right hand side shall not be negative.
This will be so when either $x \geq 1$ or $-1 \leq x \leq 0$.

5.9.4 THE FORM OF THE CURVE NEAR THE ORIGIN

For small values of x (or y), the higher powers of the variable can be neglected to give a rough idea of the shape of the curve near to the origin.

ILLUSTRATION

The curve whose equation is

$$y = 3x^3 - 2x$$

approximates to the straight line,

$$y = -2x,$$

for very small values of x.

5.9.5 ASYMPTOTES

DEFINITION

An **"asymptote"** is a straight line which is approached by a curve at a very great distance from the origin.

Asymptotes Parallel to the Co-ordinate Axes

Consider the curve whose equation is

$$y^2 = \frac{x^3(3 - 2y)}{x - 1}.$$

(a) By inspection, we see that the straight line $x = 1$ "meets" this curve at an infinite value of y, making it an asymptote parallel to the y-axis.

(b) Now re-write the equation as

$$x^3 = \frac{y^2(x-1)}{3-2y}.$$

This suggests that the straight line $y = \frac{3}{2}$ "meets" the curve at an infinite value of x, making it an asymptote parallel to the x axis.

(c) Another method for (a) and (b) is to write the equation of the curve in a form without fractions.

In this case,

$$y^2(x-1) - x^3(3-2y) = 0.$$

We then equate to zero the coefficients of the highest powers of x and y.

That is,

the coefficient of y^2 gives $x - 1 = 0$.

the coefficient of x^3 gives $3 - 2y = 0$.

This method may be used with any curve to find asymptotes parallel to the co-ordinate axes.

If there aren't any such asymptotes, the method will not work.

(ii) Asymptotes in General for a Polynomial Curve

Suppose a given curve has an equation of the form

$$P(x, y) = 0$$

where $P(x, y)$ is a polynomial in x and y.

To find the intersections with this curve of a straight line

$$y = mx + c,$$

we substitute $mx + c$ in place of y.

We obtain a polynomial equation in x, say

$$a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n = 0.$$

For the line $y = mx+c$ to be an asymptote, this equation must have **coincident solutions at infinity**.

Replace x by $\frac{1}{u}$ and multiply throughout by u^n.

$$a_0u^n + a_1u^{n-1} + a_2u^{n-2} + \ldots\ldots + a_{n-1}u + a_n = 0.$$

This equation must have coincident solutions at $u = 0$.

Hence

$$a_n = 0 \quad \text{and} \quad a_{n-1} = 0.$$

Conclusion
To find the asymptotes (if any) to a polynomial curve, we first substitute $y = mx + c$ into the equation of the curve.

Then, in the polynomial equation obtained, we **equate to zero the two leading coefficents** (that is, the coefficients of the highest two powers of x) and solve for m amd c.

EXAMPLE

Determine the equations of the asymptotes to the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solution

Substituting $y = mx + c$ gives

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1.$$

That is,

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0.$$

Equating to zero the two leading coefficients; that is, the coefficients of x^2 and x, we obtain

$$\frac{1}{a^2} - \frac{m^2}{b^2} = 0 \text{ and } \frac{2mc}{b^2} = 0.$$

No solution is obtainable if $m = 0$ in the second statement since it implies $\frac{1}{a^2} = 0$ in the first statement.

Therefore, let $c = 0$ in the second statement, and $m = \pm\frac{b}{a}$ in the first statement.

The asymptotes are therefore

$$y = \pm\frac{b}{a}x \quad \text{that is} \quad \frac{x}{a} \pm \frac{y}{b} = 0.$$