"JUST THE MATHS"

SLIDES NUMBER

5.8

GEOMETRY 8
(Conic sections - the hyperbola)

by

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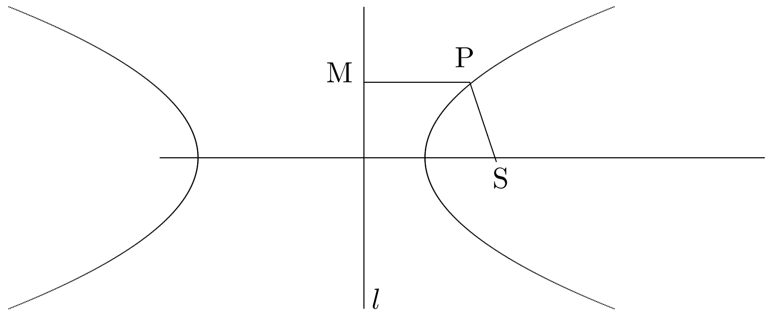
UNIT 5.8 - GEOMETRY 8

CONIC SECTIONS - THE HYPERBOLA

5.8.1 INTRODUCTION

The Standard Form for the equation of a Hyperbola



DEFINITION

The hyperbola is the path traced out by (or "locus" of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$SP = \epsilon.PM,$$

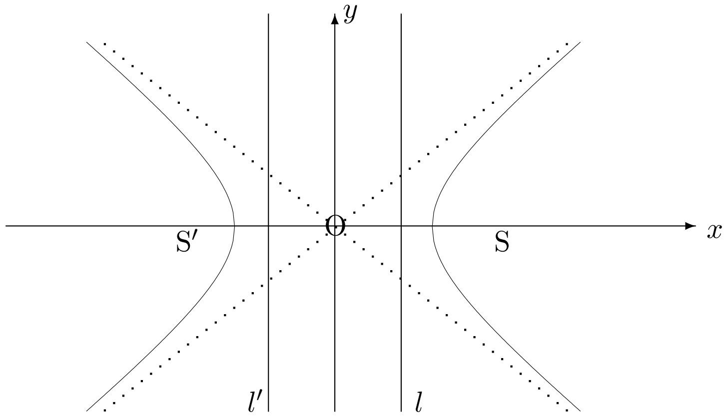
where $\epsilon > 1$ is a constant called the **"eccentricity"** of the hyperbola.

The fixed line, l, is called a **"directrix"** of the hyperbola and the fixed point, S, is called a **"focus"** of the hyperbola.

The hyperbola has two foci and two directrices because the curve is symmetrical about a line parallel to l **and** about the perpendicular line from S onto l

The following diagram illustrates two foci S and S′ together with two directrices l and l'.

The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the hyperbola has equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\sec\theta, \;\; y = b\tan\theta.$$

For students who are familiar with "hyperbolic functions", a set of parametric equations for the hyperbola is

$$x = a\text{cosh}t, \;\; y = b\text{sinh}t.$$

The curve intersects the x-axis at $(\pm a, 0)$ but does not intersect the y-axis at all.

The eccentricity, ϵ, is obtainable from the formula

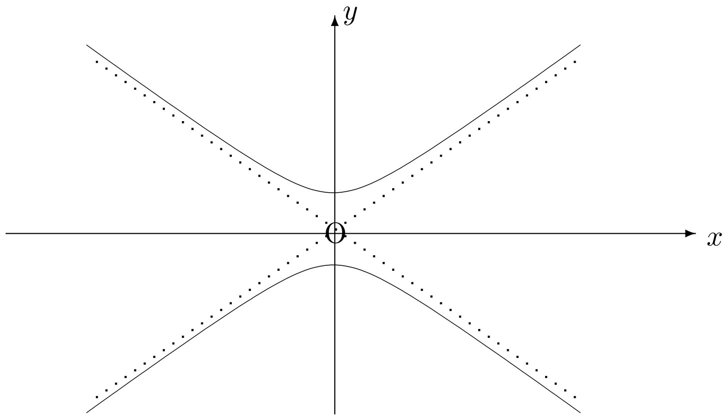
$$b^2 = a^2\left(\epsilon^2 - 1\right).$$

The foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$.

Note:
A hyperbola with centre $(0, 0)$, symmetrical about Ox and Oy, but intersecting the y-axis rather than the x-axis, has equation,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The roles of x and y are simply reversed.



5.8.2 ASYMPTOTES

At infinity, the hyperbola approaches two straight lines through the centre of the hyperbola called **"asymptotes"**.

It can be shown that both of the hyperbolae

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

have asymptotes whose equations are:

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

The equations of the asymptotes of a hyperbola are easily remembered by factorising the **left-hand side** of its equation, then equating each factor to zero.

5.8.3 MORE GENERAL FORMS FOR THE EQUATION OF A HYPERBOLA

The equation of a hyperbola, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

with associated parametric equations

$$x = h + a\text{sec}\theta, \ \ y = k + b\tan\theta$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1,$$

with associated parametric equations

$$x = h + a\tan\theta, \ \ y = k + b\text{sec}\theta.$$

Hyperbolae will usually be encountered in the expanded form of the standard cartesian equations.

It will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the hyperbola.

EXAMPLE

Determine the co-ordinates of the centre and the equations of the asymptotes of the hyperbola whose equation is

$$4x^2 - y^2 + 16x + 6y + 6 = 0.$$

Solution

Completing the square in the x terms gives

$$4x^2 + 16x \equiv 4\left[x^2 + 4x\right]$$

$$\equiv 4\left[(x+2)^2 - 4\right]$$

$$\equiv 4(x+2)^2 - 16.$$

Completing the square in the y terms gives

$$-y^2 + 6y \equiv -\left[y^2 - 6y\right]$$

$$\equiv -\left[(y-3)^2 - 9\right]$$

$$\equiv -(y-3)^2 + 9.$$

Hence the equation of the hyperbola becomes

$$4(x+2)^2 - (y-3)^2 = 1$$

or

$$\frac{(x+2)^2}{\left(\frac{1}{2}\right)^2} - \frac{(y-3)^2}{1^2} = 1.$$

The centre is located at the point $(-2, 3)$.

The asymptotes are

$$2(x+2) - (y-3) = 0 \text{ and } 2(x+2) + (y-3) = 0.$$

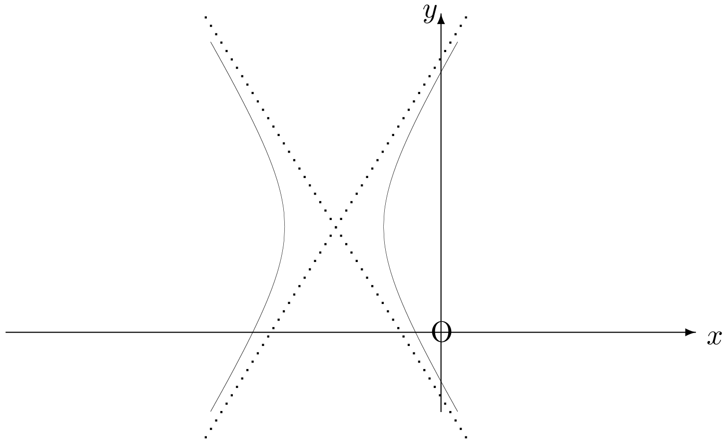
In other words,

$$2x - y + 7 = 0 \text{ and } 2x + y + 1 = 0.$$

To sketch the graph of a hyperbola, it is not always enough to have the position of the centre and the equations of the asymptotes.

It may also be necessary to investigate some of the intersections of the curve with the co-ordinate axes.

In the current example, it is possible to determine intersections at $(-0.84, 0)$, $(-7.16, 0)$, $(0, -0.87)$ and $(0, 6.87)$.



5.8.4 THE RECTANGULAR HYPERBOLA

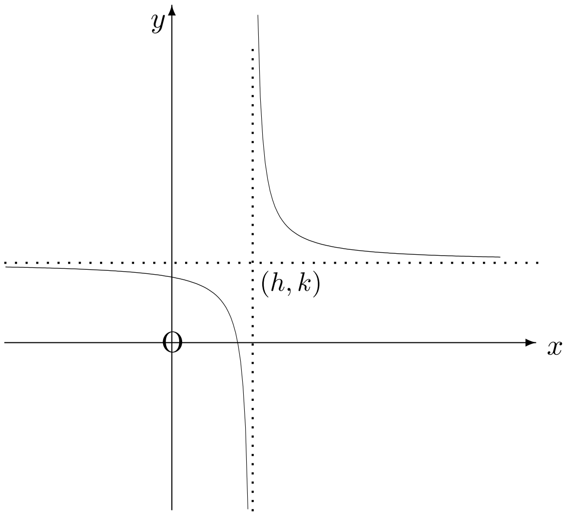
For some hyperbolae, the asymptotes are at right-angles to each other.

In this case, the **asymptotes themselves** could be used as the x-axis and y-axis.

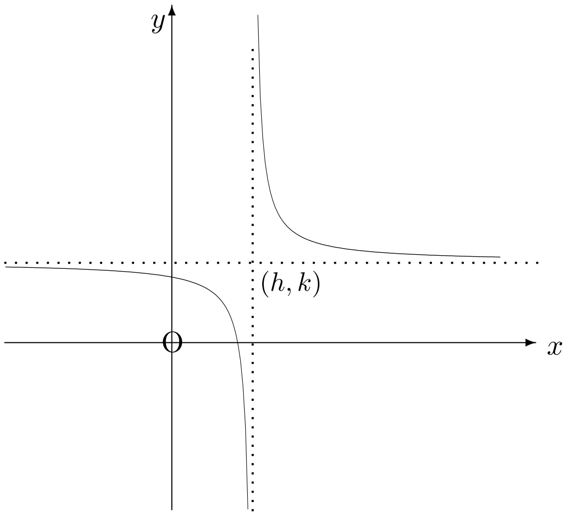
When this choice of reference system, the hyperbola, centre $(0, 0)$, has the equation

$$xy = C,$$

where C is a constant.



Similarly, a rectangular hyperbola with centre at the point (h, k) and asymptotes used as the axes of reference, has the equation, $(x - h)(y - k) = C$.



Note:

A suitable pair of parametric equations for the rectangular hyperbola, $(x - h)(y - k) = C$, are

$$x = t + h, \quad y = k + \frac{C}{t}.$$

EXAMPLES

1. Determine the centre of the rectangular hyperbola whose equation is

$$7x - 3y + xy - 31 = 0.$$

Solution

The equation factorises into the form

$$(x - 3)(y + 7) = 10.$$

Hence, the centre is located at the point $(3, -7)$.

2. A certain rectangular hyperbola has parametric equations,

$$x = 1 + t, \quad y = 3 - \frac{1}{t}.$$

Determine its points of intersection with the straight line $x + y = 4$.

Solution

Substituting for x and y into the equation of the straight line, we obtain

$$1 + t + 3 - \frac{1}{t} = 4 \text{ or } t^2 - 1 = 0.$$

Hence, $t = \pm 1$ giving points of intersection at $(2, 2)$ and $(0, 4)$.