"JUST THE MATHS"

SLIDES NUMBER

5.7

GEOMETRY 7
(Conic sections - the ellipse

by

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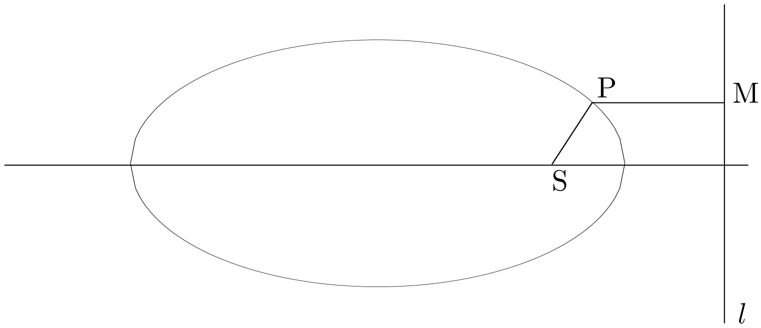
UNIT 5.7 - GEOMETRY 7

CONIC SECTIONS - THE ELLIPSE

5.7.1 INTRODUCTION

The Standard Form for the equation of an Ellipse



DEFINITION

The Ellipse is the path traced out by (or **"locus"** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

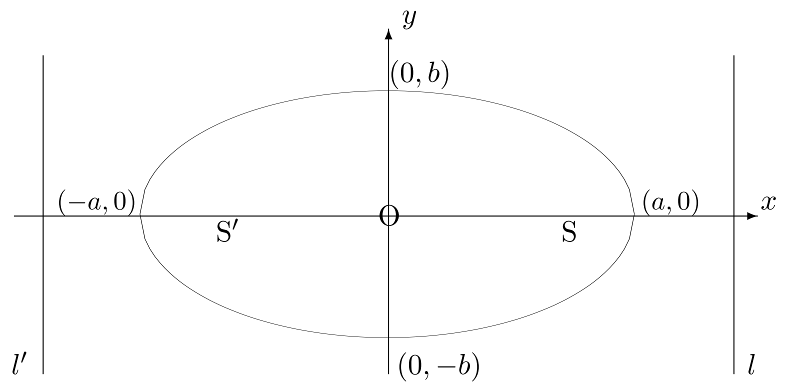
where $\epsilon < 1$ is a constant called the **"eccentricity"** of the ellipse.

The fixed line, l, is called a **"directrix"** of the ellipse and the fixed point, S, is called a **"focus"** of the ellipse.

The ellipse has two foci and two directrices because the curve is symmetrical about a line parallel to l **and** about the perpendicular line from S onto l.

The following diagram illustrates two foci, S and S$'$, together with two directrices, l and l'.

The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with associated parametric equations

$$x = a\cos\theta, \;\; y = b\sin\theta.$$

The curve intersects the axes at $(\pm a, 0)$ and $(0, \pm b)$.

The larger of a and b defines the length of the **"semi-major axis"**.

The smaller of a and b defines the length of the **"semi-minor axis"**.

The eccentricity, ϵ, is obtainable from the formula

$$b^2 = a^2\left(1 - \epsilon^2\right).$$

The foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$.

5.7.2 A MORE GENERAL FORM FOR THE EQUATION OF AN ELLIPSE

The equation of an ellipse, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with associated parametric equations

$$x = h + a\cos\theta,\ \ y = k + b\sin\theta.$$

Ellipses will usually be encountered in the **expanded** form of the standard cartesian equation.

It will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the ellipse.

EXAMPLE

Determine the co-ordinates of the centre and the lengths of the semi-axes of the ellipse whose equation is

$$3x^2 + y^2 + 12x - 2y + 1 = 0.$$

Solution

Completing the square in the x terms gives

$$3x^2 + 12x \equiv 3\left[x^2 + 4x\right]$$

$$\equiv 3\left[(x+2)^2 - 4\right]$$

$$\equiv 3(x+2)^2 - 12.$$

Completing the square in the y terms gives

$$y^2 - 2y \equiv (y-1)^2 - 1.$$

Hence, the equation of the ellipse becomes

$$3(x+2)^2 + (y-1)^2 = 12.$$

That is,

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{12} = 1.$$

The centre is at $(-2, 1)$ and the semi-axes have lengths $a = 2$ and $b = \sqrt{12}$.