

“JUST THE MATHS”

SLIDES NUMBER

5.6

GEOMETRY 6

(Conic sections - the parabola)

by

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5.6.1 Introduction (the standard parabola)

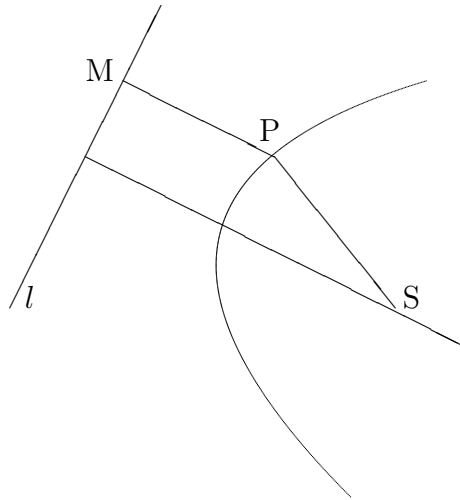
5.6.2 Other forms of the equation of a parabola

UNIT 5.6 - GEOMETRY 6

CONIC SECTIONS - THE PARABOLA

5.6.1 INTRODUCTION

The Standard Form for the equation of a Parabola



DEFINITION

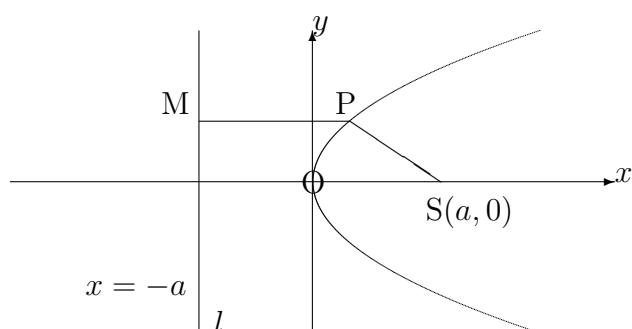
A parabola is the path traced out by (or “**locus**” of) a point, P , whose distance, SP , from a fixed point, S , called the “**focus**”, is equal to its perpendicular distance, PM , from a fixed line, l , called the “**directrix**”.

For convenience, we take the directrix l to be a vertical line, with the perpendicular onto it from the focus, S , being the x -axis.

For the y -axis, we take the line parallel to the directrix passing through the mid-point of the perpendicular from the focus onto the directrix.

The mid-point of the perpendicular from the focus onto the directrix is one of the points on the parabola.

Hence, the curve passes through the origin.



Let the focus be the point $(a, 0)$.

$SP = PM$ gives

$$\sqrt{(x - a)^2 + y^2} = x + a.$$

Squaring both sides

$$(x - a)^2 + y^2 = x^2 + 2ax + a^2,$$

or

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2.$$

This reduces to

$$y^2 = 4ax.$$

All other versions of the equation of a parabola will be based on this version.

Notes:

(i) If a is negative, the bowl of the parabola faces in the opposite direction towards negative x values.

(ii) Any equation of the form $y^2 = kx$, where k is a constant, represents a parabola with vertex at the origin and axis of symmetry along the x -axis.

Its focus will lie at the point $(\frac{k}{4}, 0)$.

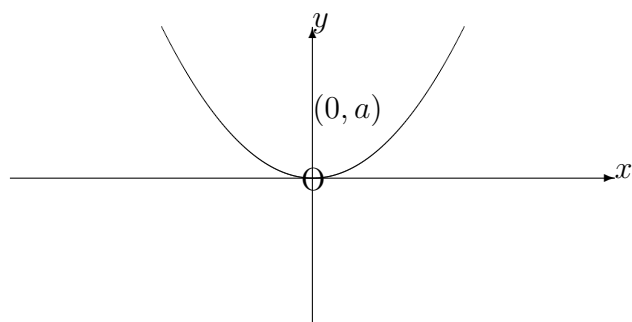
(iii) The parabola $y^2 = 4ax$ may be represented parametrically by the pair of equations

$$x = at^2, \quad y = 2at.$$

The parameter, t , has no significance in the diagram.

5.6.2 OTHER FORMS OF THE EQUATION OF A PARABOLA

(a) Vertex at $(0, 0)$ with focus at $(0, a)$



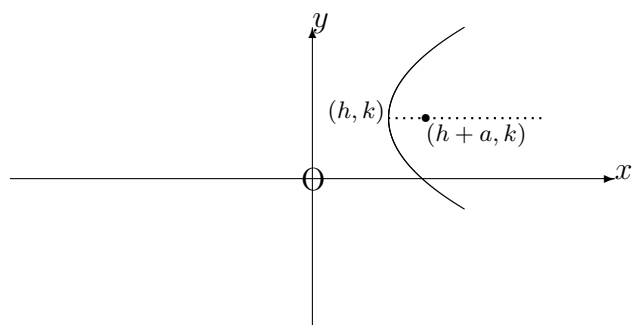
The equation is

$$x^2 = 4ay.$$

The parametric equations are

$$x = 2at, \quad y = at^2.$$

(b) Vertex at (h, k) with focus at $(h + a, k)$



Using a temporary change of origin to the point (h, k) , with X -axis and Y -axis, the parabola would have equation

$$Y^2 = 4aX.$$

With reference to the original axes, the parabola has equation

$$(y - k)^2 = 4a(x - h).$$

Notes:

(i) In the expanded form of this equation we may complete the square in the y terms to identify the vertex and focus.

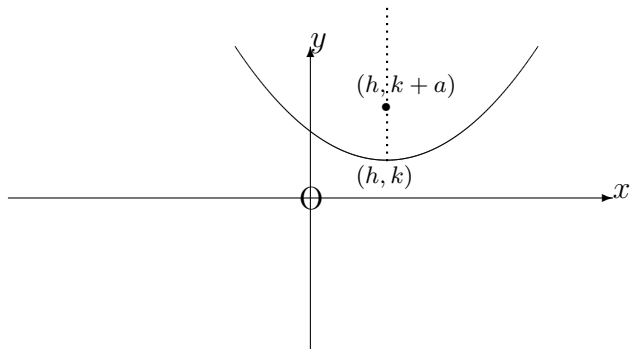
(ii) With reference to the new axes with origin at the point (h, k) , the parametric equations of the parabola would be

$$X = at^2, \quad Y = 2at.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + at^2, \quad y = k + 2at.$$

(c) Vertex at (h, k) with focus at $(h, k + a)$



Using a temporary change of origin to the point (h, k) with X -axis and Y -axis, the parabola would have equation

$$X^2 = 4aY.$$

With reference to the original axes, the parabola has equation

$$(x - h)^2 = 4a(y - k).$$

Notes:

(i) In the expanded form of this equation, we may complete the square in the x terms to identify the vertex and focus.

(ii) With reference to the new axes with origin at the point (h, k) , the parametric equations of the parabola would be

$$X = 2at, \quad Y = at^2.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + 2at, \quad y = k + at^2.$$

EXAMPLES

1. Give a sketch of the parabola whose cartesian equation is

$$y^2 - 6y + 3x = 10,$$

showing the co-ordinates of the vertex, focus and intersections with the x -axis and y -axis.

Solution

First, we complete the square in the y terms, obtaining

$$y^2 - 6y \equiv (y - 3)^2 - 9.$$

Hence, the equation becomes

$$(y - 3)^2 - 9 + 3x = 10.$$

That is,

$$(y - 3)^2 = 19 - 3x,$$

or

$$(y - 3)^2 = 4 \left(-\frac{3}{4} \right) \left(x - \frac{19}{3} \right).$$

Thus, the vertex lies at the point $\left(\frac{19}{3}, 3\right)$ and the focus lies at the point $\left(\frac{19}{3} - \frac{3}{4}, 3\right)$; that is, $\left(\frac{67}{12}, 3\right)$.

The parabola $y^2 - 6y + 3x = 10$ intersects the x -axis where $y = 0$; that is,

$$3x = 10,$$

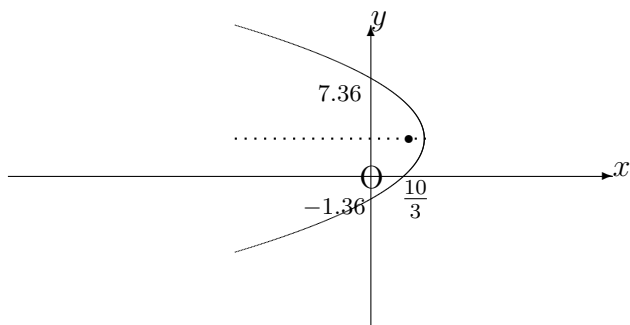
giving $x = \frac{10}{3}$.

The parabola $y^2 - 6y + 3x = 10$ intersects the y -axis where $x = 0$; that is,

$$y^2 - 6y - 10 = 0.$$

This is a quadratic equation with solutions

$$y = \frac{6 \pm \sqrt{36 + 40}}{2} \cong 7.36 \text{ or } -1.36$$



2. Use the parametric equations of the parabola

$$x^2 = 8y$$

to determine its points of intersection with the straight line

$$y = x + 6.$$

Solution

The parametric equations are $x = 4t$, $y = 2t^2$.

Substituting the parametric equations into the equation of the straight line,

$$2t^2 = 4t + 6.$$

That is,

$$t^2 - 2t - 3 = 0,$$

or

$$(t - 3)(t + 1) = 0,$$

which is a quadratic equation in t having solutions $t = 3$ and $t = -1$.

The points of intersection are therefore $(12, 18)$ and $(-4, 2)$.