"JUST THE MATHS"

SLIDES NUMBER

5.5

GEOMETRY 5
(Conic sections - the circle)

by

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UNIT 5.5 - GEOMETRY 4

CONIC SECTIONS - THE CIRCLE

5.5.1 INTRODUCTION

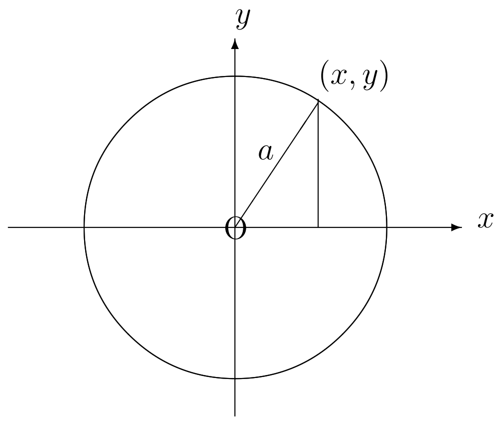
The Circle, the Parabola, the Ellipse and the Hyperbola could be generated, if desired, by considering plane sections through a cone; and, because of this, they are often called **"conic sections"** or even just **"conics"**.

DEFINITION

A circle is the path traced out by (or **"locus"** of) a point which moves at a fixed distance, called the **"radius"**, from a fixed point, called the **"centre"**.

5.5.2 STANDARD EQUATIONS FOR A CIRCLE

(a) Circle with centre at the origin and having radius a.



Using Pythagoras's Theorem in the diagram,

$$x^2 + y^2 = a^2.$$

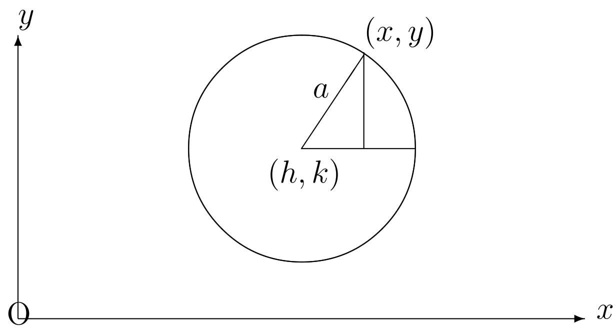
Note:
The angle θ in the diagram could be used as a parameter for the point (x, y) to give the parametric equations,

$$x = a\cos\theta, \;\; y = a\sin\theta.$$

Each point on the curve has infinitely many possible parameter values, all differing by a multiple of 2π.

We usually choose $-\pi < \theta \leq \pi$.

(b) Circle with centre (h, k) having radius a.



Using a temporary change of origin to the point (h, k) with X-axis and Y-axis, the circle would have equation

$$X^2 + Y^2 = a^2,$$

with reference to the new axes.

But, from previous work,

$$X = x - h, \text{ and } Y = y - k.$$

Hence, with reference to the original axes, the circle has equation

$$(x - h)^2 + (y - k)^2 = a^2;$$

or, in its expanded form,

$$x^2 + y^2 - 2hx - 2ky + c = 0,$$

where

$$c = h^2 + k^2 - a^2.$$

Notes:

(i) The parametric equations of this circle with reference to the temporary new axes would be

$$X = a\cos\theta,\ \ Y = a\sin\theta.$$

Hence, the parametric equations of the circle with reference to the original axes are

$$x = h + a\cos\theta,\ \ y = k + a\sin\theta.$$

(ii) If the equation of a circle is in the form

$$(x-h)^2 + (y-k)^2 = a^2,$$

it is easy to identify the centre, (h, k) and the radius, a.

If the equation is in its expanded form, we **complete the square in the x and y terms** in order to return to the first form.

EXAMPLES

1. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$x^2 + y^2 + 4x + 6y + 4 = 0.$$

Solution

$$x^2 + 4x \equiv (x + 2)^2 - 4.$$

$$y^2 + 6y \equiv (y + 3)^2 - 9.$$

$$(x + 2)^2 + (y + 3)^2 = 9.$$

Hence the centre is the point $(-2, -3)$ and the radius is 3.

2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$5x^2 + 5y^2 - 10x + 15y + 1 = 0.$$

Solution

Dividing throughout by the coefficient of the x^2 and y^2 terms,

$$x^2 + y^2 - 2x + 3y + \frac{1}{5} = 0.$$

$$x^2 - 2x \equiv (x-1)^2 - 1.$$

$$y^2 + 3y \equiv \left(y + \frac{3}{2}\right)^2 - \frac{9}{4}.$$

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{61}{20}.$$

Hence, the centre is the point $\left(1, -\frac{3}{2}\right)$ and the radius is $\sqrt{\frac{61}{20}} \cong 1.75$

Note:

Not every equation of the form

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

represents a circle.

For some combinations of h, k and c, the radius would not be a real number.

In fact,

$$a = \sqrt{h^2 + k^2 - c}$$

which could easily turn out to be unreal.