"JUST THE MATHS"

SLIDES NUMBER

5.2

GEOMETRY 2
(The straight line)

by

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UNIT 5.2 - GEOMETRY 2

THE STRAIGHT LINE

5.2.1 PREAMBLE

It is not possible to give a satisfactory diagramatic definition of a straight line

The attempt is likely to assume a knowledge of linear measurement which, itself, depends on the concept of a straight line

DEFINITION

A straight line is a set of points with cartesian co-ordinates (x, y) satisfying an equation of the form

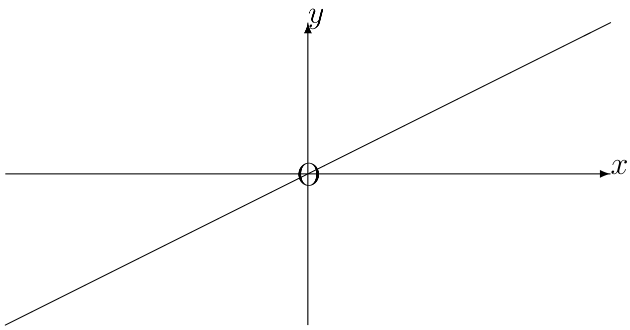
$$ax + by + c = 0,$$

where a, b and c are constants

This equation is called a **"linear equation"** and the symbol (x, y) itself, rather than a dot on the page, is an arbitrary point of the line.

5.2.2 STANDARD EQUATIONS OF A STRAIGHT LINE

(a) Having a given gradient and passing through the origin



Let the gradient be m.

All points (x, y) on the straight line
(but no others) satisfy the relationship,

$$\frac{y}{x} = m.$$

That is,

$$\boxed{y = mx.}$$

EXAMPLE

Determine, in degrees, the angle, θ, which the straight line,

$$\sqrt{3}y = x,$$

makes with the positive x-direction.

Solution

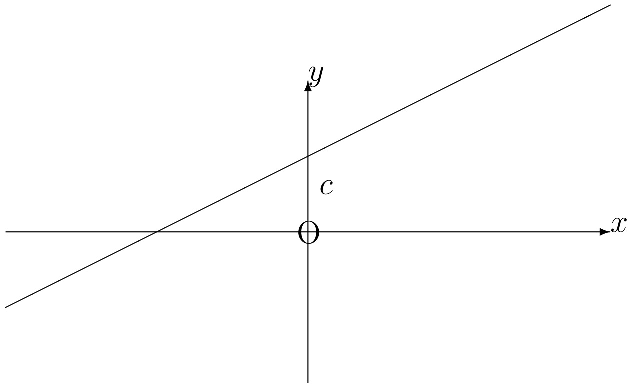
The gradient of the straight line is given by

$$\tan\theta = \frac{1}{\sqrt{3}}.$$

Hence,

$$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ.$$

(b) Having a given gradient, and a given intercept on the vertical axis



Let the gradient be m and the intercept be c.

In the previous section, add c to all of the y co-ordinates.

Hence the equation of the straight line is

$$\boxed{y = mx + c.}$$

EXAMPLE

Determine the gradient, m, and intercept c on the y-axis of the straight line whose equation is

$$7x - 5y - 3 = 0.$$

Solution

On rearranging the equation, we have

$$y = \frac{7}{5}x - \frac{3}{5}.$$

Hence,

$$m = \frac{7}{5}$$

and

$$c = -\frac{3}{5}.$$

This straight line will intersect the y-axis **below** the origin because the intercept is negative.

(c) Having a given gradient and passing through a given point

Let the gradient be m and let the given point be (x_1, y_1).

Then

$$y = mx + c,$$

where

$$y_1 = mx_1 + c.$$

Subtracting the second from the first, we obtain

$$\boxed{y - y_1 = m(x - x_1).}$$

EXAMPLE
Determine the equation of the straight line having gradient $\frac{3}{8}$ and passing through the point $(-7, 2)$.

Solution

From the formula,

$$y - 2 = \frac{3}{8}(x + 7).$$

That is,

$$8y - 16 = 3x + 21 \quad \text{or} \quad 8y = 3x + 37.$$

(d) Passing through two given points

Let the two given points be (x_1, y_1) and (x_2, y_2).

Then, the gradient is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

more usually written

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Note:
The same result is obtained no matter which way round the given points are taken as (x_1, y_1) and (x_2, y_2).

EXAMPLE

Determine the equation of the straight line joining the two points $(-5, 3)$ and $(2, -7)$, stating the values of its gradient and its intercept on the y-axis.

Solution

Method 1.

$$\frac{y-3}{-7-3} = \frac{x+5}{2+5},$$

giving

$$7(y-3) = -10(x+5) \quad \text{or} \quad 10x + 7y + 29 = 0.$$

Method 2.

$$\frac{y+7}{3+7} = \frac{x-2}{-5-2},$$

giving

$$-7(y+7) = 10(x-2) \quad \text{or} \quad 10x + 7y + 29 = 0.$$

Rewriting the equation as

$$y = -\frac{10}{7}x - \frac{29}{7},$$

the gradient is $-\frac{10}{7}$ and the intercept is $-\frac{29}{7}$.

(e) The parametric equations of a straight line

In the previous section, the common value of the two fractions

$$\frac{y - y_1}{y_2 - y_1} \quad \text{and} \quad \frac{x - x_1}{x_2 - x_1}$$

is called the **"parameter"** of the point (x, y) and is usually denoted by t.

Equating each fraction separately to t

$$x = x_1 + (x_2 - x_1)t \quad \text{and} \quad y = y_1 + (y_2 - y_1)t.$$

These are called the **"parametric equations"** of the straight line

(x_1, y_1) and (x_2, y_2) are known as the **"base points"** of the parametric representation of the line.

Notes:

(i) In the above parametric representation, (x_1, y_1) has parameter $t = 0$ and (x_2, y_2) has parameter
$t = 1$.

(ii) Other parametric representations of the same line can be found by using the given base points in the opposite order, or by using a different pair of points on the line as base points.

EXAMPLES

1. Use parametric equations to find two other points on the line joining $(3, -6)$ and $(-1, 4)$.

   Solution

   One possible parametric representation of the line is

   $$x = 3 - 4t \quad y = -6 + 10t.$$

   To find another two points, substitute any two values of t other than 0 or 1.

   For example, with $t = 2$ and $t = 3$,

   $$x = -5, y = 14 \quad \text{and} \quad x = -9, y = 24.$$

   A pair of suitable points is therefore $(-5, 14)$ and $(-9, 24)$.

2. The co-ordinates, x and y of a moving particle are given, at time t, by the equations

$$x = 3 - 4t \quad \text{and} \quad y = 5 + 2t.$$

Determine the gradient of the straight line along which the particle moves.

Solution

Eliminating t, we have

$$\frac{x-3}{-4} = \frac{y-5}{2}.$$

That is,

$$2(x-3) = -4(y-5),$$

giving

$$y = -\frac{2}{4}x + \frac{26}{4}.$$

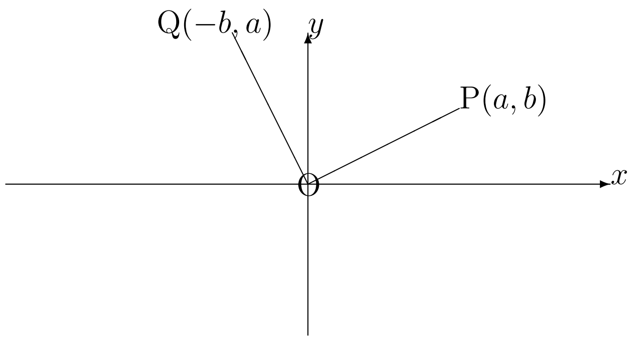
Hence, the gradient of the line is

$$-\frac{2}{4} = -\frac{1}{2}.$$

5.2.3 PERPENDICULAR STRAIGHT LINES

The perpendicularity of two straight lines is not dependent on either their length or their
precise position in the plane.

We consider two straight line segments of equal length passing through the origin.



In the diagram, the gradient of OP $= \frac{b}{a}$ and the gradient of OQ $= \frac{a}{-b}$.

Hence,
the product of the gradients is equal to -1

or

Each gradient is minus the reciprocal of the other gradient.

EXAMPLE

Determine the equation of the straight line which passes through the point $(-2, 6)$ and is perpendicular to the straight line,

$$3x + 5y + 11 = 0.$$

Solution

The gradient of the given line is $-\frac{3}{5}$.

Hence, the gradient of a perpendicular line is $\frac{5}{3}$.

Thus, the required line has equation

$$y - 6 = \frac{5}{3}(x + 2),$$

giving

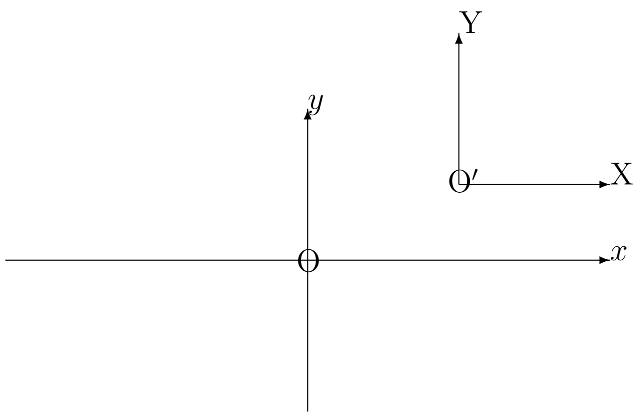
$$3y - 18 = 5x + 10.$$

That is,

$$3y = 5x + 28.$$

5.2.4 CHANGE OF ORIGIN

Given a cartesian system of reference with axes Ox and Oy, it may sometimes be convenient to consider a new set of axes O$'X$ parallel to Ox and O$'Y$ parallel to Oy with new origin at O$'$ whose co-ordinates are (h, k) referred to the original set of axes.



$$X = x - h \quad \text{and} \quad Y = y - k$$

or

$$x = X + h \quad \text{and} \quad y = Y + k.$$

EXAMPLE
Given the straight line,

$$y = 3x + 11,$$

determine its equation referred to new axes with new origin at the point $(-2, 5)$.

Solution

Using

$$x = X - 2 \quad \text{and} \quad y = Y + 5$$

we obtain

$$Y + 5 = 3(X - 2) + 11.$$

That is,

$$Y = 3X,$$

which is a straight line through the new origin with gradient 3.

Note:
The point $(-2, 5)$ is **on** the original line

Hence, the new line passes through the new origin.

Its gradient would not alter in the change of origin.