"JUST THE MATHS"

UNIT NUMBER

5.11

GEOMETRY 11
(Polar curves)

by

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UNIT 5.11 - GEOMETRY 11 - POLAR CURVES

5.11.1 INTRODUCTION

For conversion from cartesian co-ordinates, x and y, to polar co-ordinates, r and θ, we use the formulae,

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta,$$

For the reverse process, we may use the formulae,

$$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

Sometimes the reverse process may be simplified by using a mixture of both sets of formulae.

We shall consider the graphs of certain relationships between r and θ without necessarily refering to the equivalent of those relationships in cartesian co-ordinates.

The graphs obtained will be called **"polar curves"**.

Note:

For the present context it will be necessary to assign a meaning to a point (r, θ), in polar co-ordinates, when r is negative.

We plot the point at a distance of $|r|$ along the $\theta - 180^\circ$ line.

This implies that, when r is negative, the point (r, θ) is the same as the point $(|r|, \theta - 180^\circ)$

5.11 2 THE USE OF POLAR GRAPH PAPER

For equations in which r is expressed in terms of θ, we plot r against θ using a graph paper divided into small cells by concentric circles and radial lines.

The radial lines are usually spaced at intervals of 15°.

The concentric circles allow a scale to be chosen to measure the distances, r, from the pole.

EXAMPLES

1. Sketch the graph of the equation $r = 2\sin\theta$.

   Solution

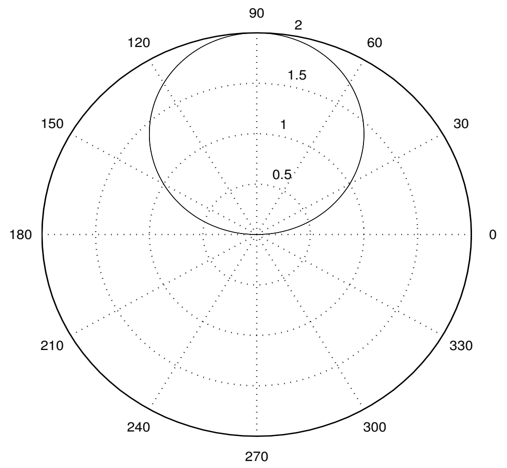
   First we construct a table of values of r and θ, in steps of 15°, from 0° to 360°.

θ	0°	15°	30°	45°	60°	75°	90°
r	0	0.52	1	1.41	1.73	1.93	2

θ	105°	120°	135°	150°	165°	180°	195°
r	1.93	1.73	1.41	1	0.52	0	-0.52

θ	210°	225°	240°	255°	270°	285°
r	-1	-1.41	-1.73	-1.93	-2	-1.93

θ	300°	315°	330°	345°	360°
r	-1.73	-1.41	-1	-0.52	0



Notes:

(i) The curve is a circle whose cartesian equation turns out to be

$$x^2 + y^2 - 2y = 0.$$

(ii) Since half of the values of r are negative, the circle is described twice over.

For example, the point $(-0.52, 195^\circ)$ is the same as the point $(0.52, 15^\circ)$.

2. Sketch the graph of the following equations:

(a)

$$r = 2(1 + \cos\theta);$$

(b)

$$r = 1 + 2\cos\theta;$$

(c)

$$r = 5 + 3\cos\theta.$$

Solution

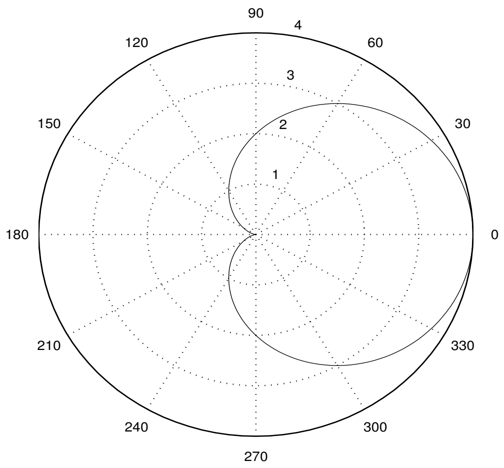
(a) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
r	4	3.93	3.73	3.42	3	2.52	2

θ	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	1.48	1	0.59	0.27	0.07	0

θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$
r	0.07	0.27	0.59	1	1.48	2

θ	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	2.52	3	3.42	3.73	3.93	4



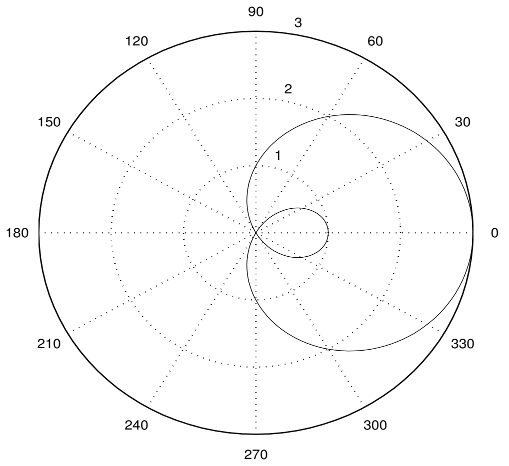
(b) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
r	3	2.93	2.73	2.41	2	1.52	1

θ	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	30.48	0	-0.41	-0.73	-0.93	-1

θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$
r	-0.93	-0.73	-0.41	0	0.48	1

θ	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	1.52	2	2.41	2.73	2.93	3



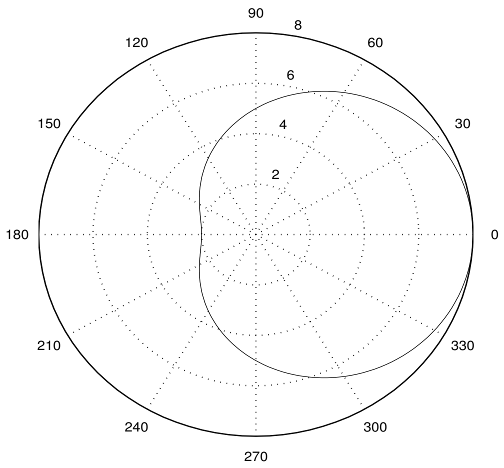
(c) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
r	8	7.90	7.60	7.12	6.5	5.78	5

θ	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	4.22	3.5	2.88	2.40	2.10	2

θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$
r	2.10	2.40	2.88	3.5	4.22	5

θ	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	5.78	6.5	7.12	7.60	7.90	8



Note:

Each of the three curves in the above example is known as a **"limacon"**.

They illustrate special cases of the more general curve, $r = a + b\cos\theta$, as follows:

(i) If $a = b$, the limacon may also be called a **"cardioid"** (heart-shape). At the pole, the curve possesses a **"cusp"**.

(ii) If $a < b$, the limacon contains a **"re-entrant loop"**.

(iii) If $a > b$, the limacon contains neither a cusp nor a re-entrant loop.

For other well-known polar curves, together with any special titles associated with them, refer to the answers to the exercises associated with this unit.