"JUST THE MATHS"

SLIDES NUMBER

5.1

GEOMETRY 1
(Co-ordinates, distance & gradient)

by

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UNIT 5.1 - GEOMETRY 1

CO-ORDINATES, DISTANCE AND GRADIENT

5.1.1 CO-ORDINATES

(a) Cartesian Co-ordinates

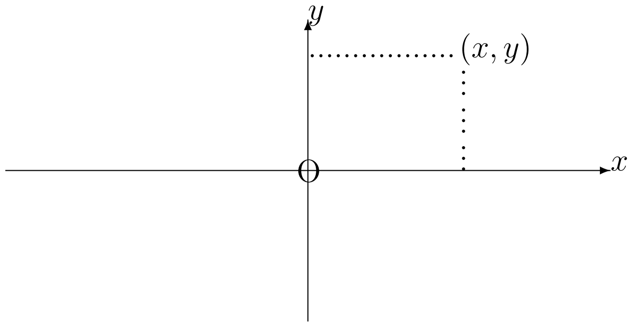
The position of a point P in a plane may be specified completely if we know its perpendicular distances from two chosen fixed straight lines

We distinguish between positive distances on one side of each line and negative distances on the other side of each line.

It is not essential that the two chosen fixed lines should be at right- angles to each other

We usually take them to be so for the sake of convenience.

Consider the following diagram:



The horizontal directed line, Ox, is called the **"x-axis"**.

Distances to the right of the origin (point O) are taken as positive.

The vertical directed line, Oy, is called the **"y-axis"**.

Distances above the origin (point O) are taken as positive.

The notation (x, y) denotes a point whose perpendicular distances from Oy and Ox are x and y respectively, the **"cartesian co-ordinates"** of the point.

(b) Polar Co-ordinates

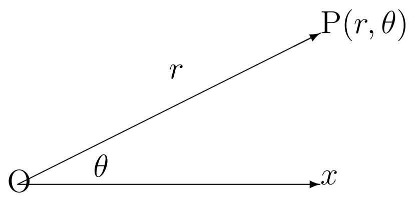
This is an alternative method of fixing the position of a point P in a plane.

First choose a point, O, called the **"pole"**.

Then choose a directed line , Ox, emanating from the pole in one direction only

Ox is called the **"initial line"**.

Consider the following diagram:



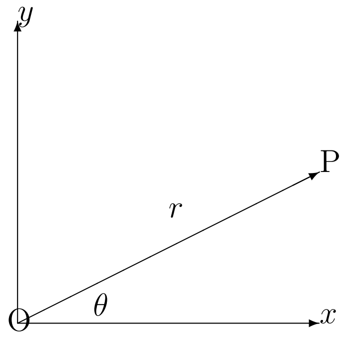
The position of P is determined by its distance r from the pole and the angle, θ which the line OP makes with the initial line.

θ is measured positively in a counter-clockwise sense or negatively in a clockwise sense from the initial line.

(r, θ) denotes the "polar co-ordinates" of the point.

5.1.2 THE RELATIONSHIP BETWEEN POLAR & CARTESIAN CO-ORDINATES

Superimpose one diagram upon the other.



(a) $x = r.\cos\theta$ and $y = r.\sin\theta$;

(b) $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

EXAMPLES

1. Express the equation

$$2x + 3y = 1$$

in polar co-ordinates.

Solution

Substituting for x and y separately,

$$2r\cos\theta + 3r\sin\theta = 1.$$

That is,

$$r = \frac{1}{2\cos\theta + 3\sin\theta}.$$

2. Express the equation

$$r = \sin\theta$$

in cartesian co-ordinates.

Solution

We could try substituting for r and θ separately; but it is easier to rewrite the equation as

$$r^2 = r\sin\theta,$$

which gives

$$x^2 + y^2 = y.$$

5.1.3 THE DISTANCE BETWEEN 2 POINTS

Given two points (x_1, y_1) and (x_2, y_2), the quantity $\mid x_2 - x_1 \mid$ is called the **"horizontal separation"** of the two points.

The quantity $\mid y_2 - y_1 \mid$ is called the **"vertical separation"** of the two points.

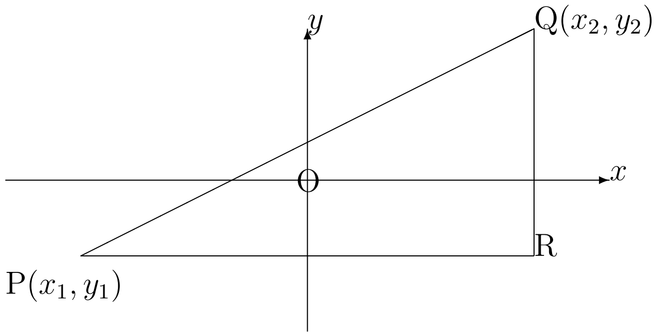
We are assuming that the x-axis is horizontal.

The expressions remain valid even when one or more of the co-ordinates is negative.

For example, the horizontal separation of the points $(5, 7)$ and $(-3, 2)$ is given by $\mid -3 - 5 \mid = 8$.

This agrees with the two points being on opposite sides of the y-axis.

The actual distance between (x_1, y_1) and (x_2, y_2) may be calculated from Pythagoras' Theorem, using their horizontal and vertical separations.



In the diagram,

$$\text{PQ}^2 = \text{PR}^2 + \text{RQ}^2.$$

That is,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

giving

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note: There is no need to include the modulus signs of the horizontal and vertical separations.

It does not matter which way round the points are labelled.

EXAMPLE
Calculate the distance, d, between the two points $(5, -3)$ and $(-11, -7)$.

Solution

$$d = \sqrt{(5 + 11)^2 + (-3 + 7)^2}.$$

That is,

$$d = \sqrt{256 + 16} = \sqrt{272} \cong 16.5$$

5.1.4 GRADIENT

The gradient of the straight-line segment PQ is defined to be the tangent of the angle which PQ makes with the positive x-direction.

When the co-ordinates of the two points are $\mathrm{P}(x_1, y_1)$ and $\mathrm{Q}(x_2, y_2)$, the gradient, m, is given by either

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

We distinguish between positive & negative gradient.

EXAMPLE
Obtain the gradient of the straight-line segment joining the two points $(8, -13)$ and $(-2, 5)$ and hence calculate the angle, θ, which the segment makes with the positive x-direction.

Solution

$$m = \frac{5 + 13}{-2 - 8} = \frac{-13 - 5}{8 + 2} = -1.8$$

Hence,

$$\tan\theta = -1.8$$

giving

$$\theta = \tan^{-1}(-1.8) \simeq 119^\circ.$$