"JUST THE MATHS"

SLIDES NUMBER

3.5

TRIGONOMETRY 5
(Trigonometric identities & wave-forms)

by

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UNIT 3.5 - TRIGONOMETRY 5 TRIGONOMETRIC IDENTITIES AND WAVE-FORMS

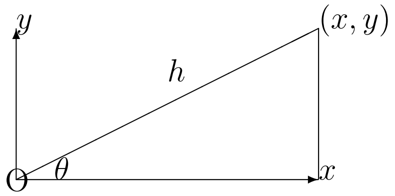
3.5.1 TRIGONOMETRIC IDENTITIES

ILLUSTRATION

Prove that

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

Proof:



$$\cos\theta = \frac{x}{h} \quad \text{and} \quad \sin\theta = \frac{y}{h};$$

$$x^2 + y^2 = h^2;$$

$$\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 = 1;$$

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

Other Variations

(a) $\cos^2\theta \equiv 1 - \sin^2\theta$; (rearrangement).

(b) $\sin^2\theta \equiv 1 - \cos^2\theta$; (rearrangement).

(c) $\sec^2\theta \equiv 1 + \tan^2\theta$; (divide by $\cos^2\theta$).

(d) $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$; (divide by $\sin^2\theta$).

Other Trigonometric Identities

$$\sec\theta \equiv \frac{1}{\cos\theta}$$

$$\text{cosec}\theta \equiv \frac{1}{\sin\theta}$$

$$\cot\theta \equiv \frac{1}{\tan\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1$$

$$1 + \tan^2\theta \equiv \sec^2\theta$$

$$1 + \cot^2\theta \equiv \text{cosec}^2\theta$$

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A$$

$$\equiv 1 - 2\sin^2 A$$

$$\equiv 2\cos^2 A - 1$$

$$\sin A \equiv 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$\cos A \equiv \cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A$$

$$\equiv 1 - 2\sin^2 \frac{1}{2} A$$

$$\equiv 2\cos^2 \frac{1}{2} A - 1$$

$$\tan A \equiv \frac{2 \tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A}$$

$$\sin A + \sin B \equiv 2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B \equiv 2 \cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B \equiv 2 \cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B \equiv -2 \sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)$$

$$\sin A\cos B \equiv \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$
$$\cos A\sin B \equiv \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$$
$$\cos A\cos B \equiv \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$$
$$\sin A\sin B \equiv \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$
$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$
$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

EXAMPLES

1. Show that
$$\sin^2 2x \equiv \frac{1}{2}(1-\cos 4x).$$

Solution

$$\cos 4x \equiv 1 - 2\sin^2 2x.$$

2. Show that
$$\sin\left(\theta+\frac{\pi}{2}\right) \equiv \cos\theta.$$

Solution

The left hand side can be expanded as
$$\sin\theta\cos\frac{\pi}{2}+\cos\theta\sin\frac{\pi}{2};$$
The result follows, because $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$.

3. Simplify the expression

$$\frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha}.$$

Solution

Expression becomes

$$\frac{2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\cos\left(\frac{2\alpha-3\alpha}{2}\right)}{-2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\sin\left(\frac{2\alpha-3\alpha}{2}\right)}$$

$$\equiv \frac{2\sin\left(\frac{5\alpha}{2}\right).\cos\left(\frac{-\alpha}{2}\right)}{-2\sin\left(\frac{5\alpha}{2}\right).\sin\left(\frac{-\alpha}{2}\right)}$$

$$\equiv \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\equiv \cot\left(\frac{\alpha}{2}\right).$$

4. Express $2\sin 3x \cos 7x$ as the difference of two sines.

Solution

$$2\sin 3x \cos 7x \equiv \sin(3x+7x) + \sin(3x-7x).$$

Hence,

$$2\sin 3x \cos 7x \equiv \sin 10x - \sin 4x.$$

3.5.2 AMPLITUDE, WAVE-LENGTH, FREQUENCY AND PHASE ANGLE

Importance is attached to trigonometric functions of the form

$$A\sin(\omega t+\alpha) \quad \text{and} \quad A\cos(\omega t+\alpha),$$

where A, ω and α are constants and t is usually a time variable.

The expanded forms are

$$A\sin(\omega t+\alpha)\equiv A\sin\omega t\cos\alpha + A\cos\omega t\sin\alpha$$

and

$$A\cos(\omega t+\alpha)\equiv A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha.$$

(a) The Amplitude

A, represents the maximum value (numerically) which can be attained by each of the above trigonometric functions.

A is called the **"amplitude"** of each of the functions.

(b) The Wave Length (Or Period)

If t increases or decreases by a whole multiple of $\frac{2\pi}{\omega}$, then $(\omega t + \alpha)$ increases or decreases by a whole multiple of 2π; and hence the functions remain unchanged in value.

A graph, against t, of either $A\sin(\omega t + \alpha)$ or $A\cos(\omega t + \alpha)$ would be repeated in shape at regular intervals of length $\frac{2\pi}{\omega}$.

The repeated shape of the graph is called the **"wave profile"** and $\frac{2\pi}{\omega}$ is called the **"wave-length"**, or **"period"** of each of the functions.

(c) The Frequency

If t is a time variable, then the wave length (or period) represents the time taken to complete a single wave-profile.

Consequently, the number of wave-profiles completed in one unit of time is given by $\frac{\omega}{2\pi}$.

$\frac{\omega}{2\pi}$ is called the **"frequency"** of each of the functions.

Note:
ω is called the **"angular frequency"**;

ω represents the change in the quantity $(\omega t+\alpha)$ for every unit of change in the value of t.

(d) The Phase Angle

α affects the starting value, at $t = 0$, of the trigonometric functions $A\sin(\omega t + \alpha)$ and $A\cos(\omega t + \alpha)$.

Each of these is said to be **"out of phase"**, by an amount, α, with the trigonometric functions $A\sin\omega t$ and $A\cos\omega t$ respectively.

α is called the **"phase angle"** of each of the two original trigonometric functions;

it can take infinitely many values differing only by a whole multiple of $360°$ or 2π.

EXAMPLES

1. Express $\sin t + \sqrt{3}\cos t$ in the form $A\sin(t + \alpha)$, with α in degrees, and hence solve the equation,

   $$\sin t + \sqrt{3}\cos t = 1,$$

   for t in the range $0° \leq t \leq 360°$.

   Solution

   We require that

   $$\sin t + \sqrt{3}\cos t \equiv A\sin t\cos\alpha + A\cos t\sin\alpha$$

Hence,

$$A\cos\alpha = 1 \;\text{ and }\; A\sin\alpha = \sqrt{3},$$

which gives $A^2 = 4$ (using $\cos^2\alpha + \sin^2\alpha \equiv 1$) and also $\tan\alpha = \sqrt{3}$.

Thus,

$$A = 2 \;\text{ and }\; \alpha = 60^\circ \text{ (principal value)}.$$

To solve the given equation, we may now use

$$2\sin(t + 60^\circ) = 1,$$

so that

$$t + 60^\circ = \text{Sin}^{-1}\frac{1}{2} = 30^\circ + k360^\circ \;\text{ or }\; 150^\circ + k360^\circ,$$

where k may be any integer.

For the range $0^\circ \leq t \leq 360^\circ$, we conclude that

$$t = 330^\circ \;\text{ or }\; 90^\circ.$$

2. Express $a\sin\omega t + b\cos\omega t$ in the form $A\sin(\omega t+\alpha)$. Apply the result to the expression $3\sin 5t - 4\cos 5t$ stating α in degrees, correct to one decimal place, and lying in the interval from $-180°$ to $180°$.

Solution

$$A\sin(\omega t+\alpha) \equiv a\sin\omega t + b\cos\omega t;$$

$$A\sin\alpha = b \quad \text{and} \quad A\cos\alpha = a;$$

$$A^2 = a^2 + b^2;$$

$$A = \sqrt{a^2+b^2}.$$

Also

$$\frac{A\sin\alpha}{A\cos\alpha} = \frac{b}{a};$$

$$\alpha = \tan^{-1}\frac{b}{a}.$$

Note:

The particular angle chosen must ensure that $\sin\alpha = \frac{b}{A}$ and $\cos\alpha = \frac{a}{A}$ have the correct sign.

For $3\sin 5t - 4\cos 5t$, we have

$$A = \sqrt{3^2 + 4^2}$$

and

$$\alpha = \tan^{-1}\left(-\frac{4}{3}\right).$$

But $\sin\alpha \left(= -\frac{4}{5}\right)$ and $\cos\alpha \left(= \frac{3}{5}\right)$ so that $-90^\circ < \alpha < 0$;

that is $\alpha = -53.1^\circ$.

We conclude that

$$3\sin 5t - 4\cos 5t \equiv 5\sin(5t - 53.1^\circ)$$

3. Solve the equation

$$4\sin 2t + 3\cos 2t = 1$$

for t in the interval from $-180°$ to $180°$.

Solution

Expressing the left hand side of the equation in the form $A\sin(2t+\alpha)$, we require

$$A = \sqrt{4^2+3^2} = 5 \quad \text{and} \quad \alpha = \tan^{-1}\frac{3}{4}.$$

Also $\sin\alpha\left(=\frac{3}{5}\right)$ and $\cos\alpha\left(=\frac{4}{5}\right)$ so that $0 < \alpha < 90°$.

Hence, $\alpha = 36.87°$ and

$$5\sin(2t + 36.87°) = 1.$$

$$t = \frac{1}{2}\left[\mathrm{Sin}^{-1}\frac{1}{5} - 36.87°\right].$$

$$\mathrm{Sin}^{-1}\frac{1}{5} = 11.53° + k360° \quad \text{and} \quad 168.46° + k360°,$$

where k may be any integer.

But, for t values which are numerically less than $180°$, we use $k = 0$ and $k = 1$ in the first and $k = 0$ and $k = -1$ in the second.

$$t = -12.67°,\ 65.80°,\ 167.33° \quad \text{and} \quad -114.21°$$