“JUST THE MATHS”

SLIDES NUMBER

3.4

TRIGONOMETRY 4
(Solution of triangles)

by

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UNIT 3.4 - TRIGONOMETRY 4

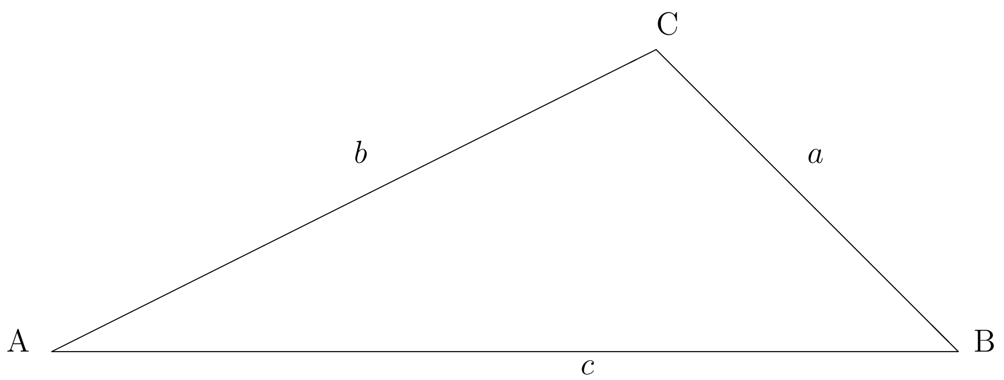
SOLUTION OF TRIANGLES

3.4.1 INTRODUCTION

The **"solution of a triangle"** is defined to mean the complete set of data relating to the lengths of its three sides and the values of its three interior angles.

It can be shown that interior angles add up to 180°.

For an arbitrary triangle with "vertices" A,B and C and sides of length a, b and c, we draw



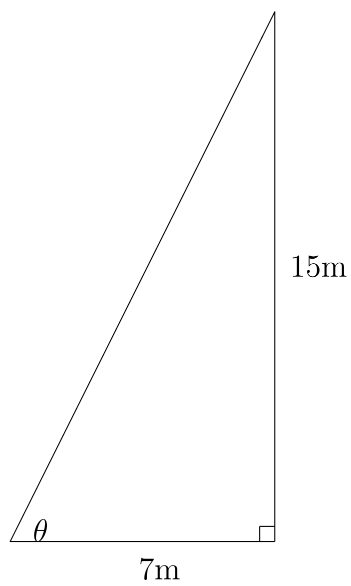
The angles at A,B and C will be denoted by $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$.

3.4.2 RIGHT-ANGLED TRIANGLES

EXAMPLE

From the top of a vertical pylon, 15 meters high, a guide cable is to be secured into the (horizontal) ground at a distance of 7 meters from the base of the pylon.
What will be the length of the cable and what will be its inclination (in degrees) to the horizontal ?

Solution



From Pythagoras' Theorem, the length of the cable will be

$$\sqrt{7^2 + 15^2} \simeq 16.55\text{m}.$$

The angle of inclination to the horizontal will be θ, where

$$\tan\theta = \frac{15}{7} \text{ and } \theta \simeq 65^\circ.$$

3.4.3 THE SINE AND COSINE RULES

(a) The Sine Rule

$$\frac{a}{\sin\widehat{A}} = \frac{b}{\sin\widehat{B}} = \frac{c}{\sin\widehat{C}}.$$

(b) The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos\widehat{A};$$

$$b^2 = c^2 + a^2 - 2ca\cos\widehat{B};$$

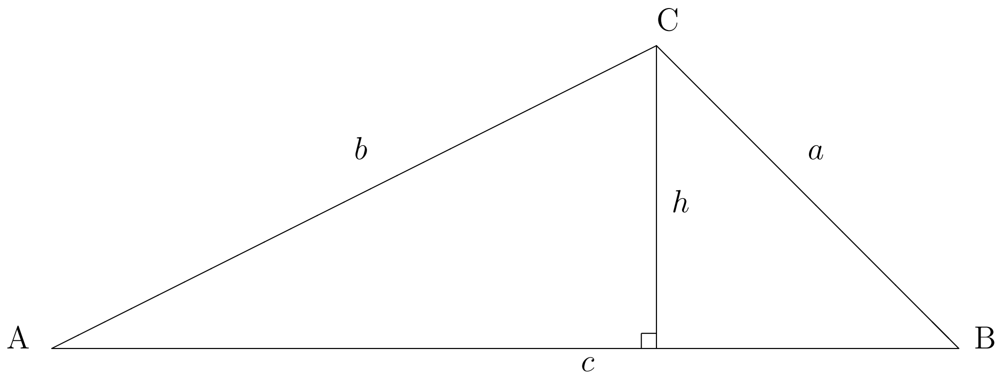
$$c^2 = a^2 + b^2 - 2ab\cos\widehat{C}.$$

Observation

Whenever the angle on the right-hand-side is a right-angle, the Cosine Rule reduces to Pythagoras' Theorem.

The Proof of the Sine Rule

First, draw the perpendicular (of length h) from the vertex C onto the side AB.



$$\frac{h}{b} = \sin \widehat{A} \quad \text{and} \quad \frac{h}{a} = \sin \widehat{B}.$$

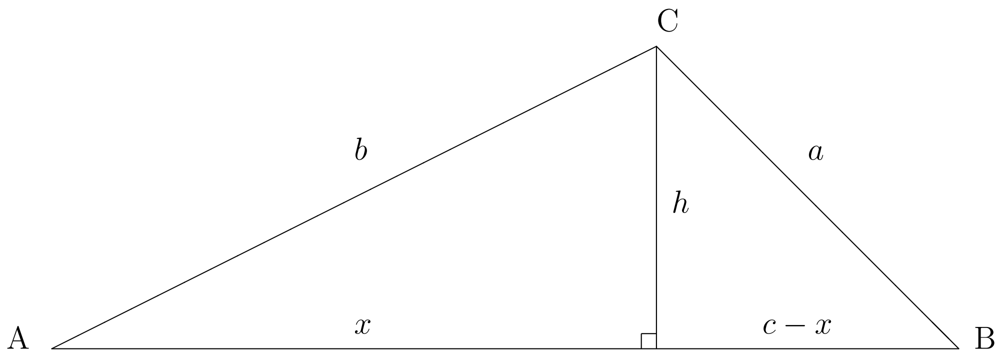
$$b \sin \widehat{A} = a \sin \widehat{B}.$$

$$\frac{b}{\sin \widehat{B}} = \frac{a}{\sin \widehat{A}}.$$

The rest of the Sine Rule can be obtained by considering the perpendicular drawn from a different vertex.

The Proof of the Cosine Rule

Let the side AB have lengths x and $c - x$ either side of the foot of the perpendicular drawn from C.



$$h^2 = b^2 - x^2$$

and

$$h^2 = a^2 - (c - x)^2.$$

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2.$$

$$a^2 = b^2 + c^2 - 2xc.$$

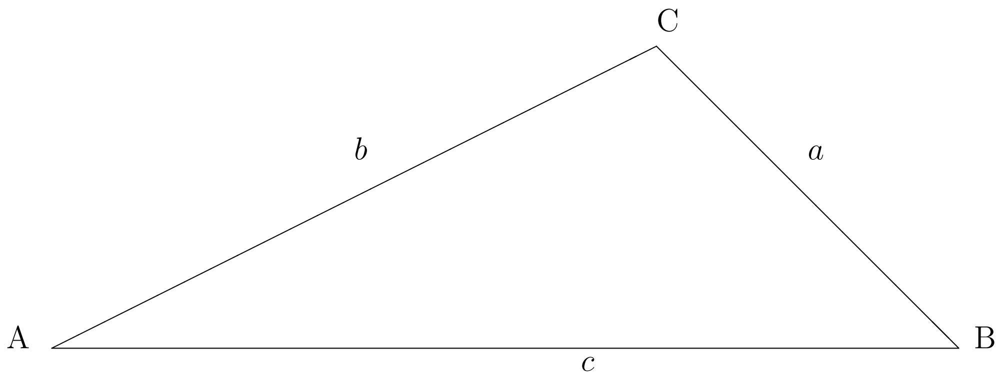
But $x = b \cos \widehat{A}$, and so

$$a^2 = b^2 + c^2 - 2bc \cos \widehat{A}.$$

EXAMPLES

1. Solve the triangle ABC in the case when
   $\widehat{A} = 20°, \widehat{B} = 30°$ and $c = 10$cm

   Solution



Firstly, the angle $\widehat{C} = 130°$ since interior angles add up to 180°.

By the Sine Rule,

$$\frac{a}{\sin 20°} = \frac{b}{\sin 30°} = \frac{10}{\sin 130°};$$

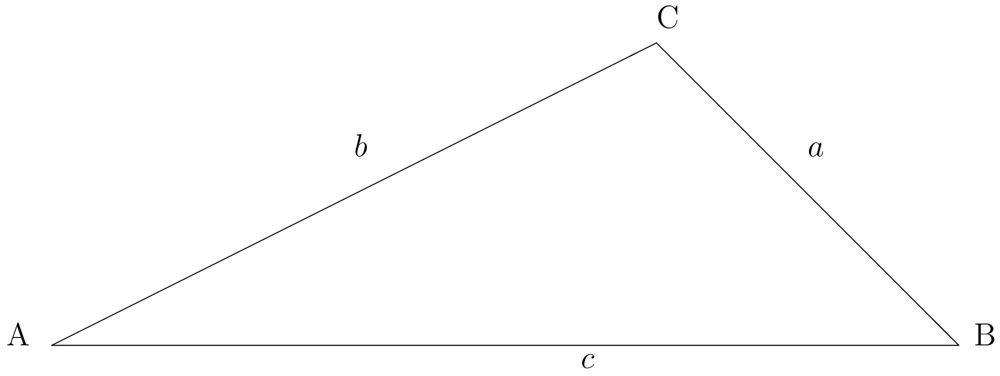
$$\frac{a}{0.342} = \frac{b}{0.5} = \frac{10}{0.766};$$

$$a = \frac{10 \times 0.342}{0.766} \cong 4.47\text{cm};$$

$$b = \frac{10 \times 0.5}{0.766} \cong 6.53\text{cm}.$$

2. Solve the triangle ABC in the case when $b = 9\text{cm}, \quad c = 5\text{cm}$ and $\widehat{A} = 70°$.

Solution



By the Cosine Rule,

$$a^2 = 25 + 81 - 90\cos 70°;$$

$$a^2 = 106 - 30.782 = 75.218;$$

$$a \simeq 8.673 \simeq 8.67\text{cm}.$$

By the Sine Rule,

$$\frac{8.673}{\sin 70°} = \frac{9}{\sin \widehat{B}} = \frac{5}{\sin \widehat{C}};$$

$$\sin \widehat{B} = \frac{9 \times \sin 70°}{8.673} = \frac{9 \times 0.940}{8.673} \simeq 0.975$$

This suggests $\widehat{B} \simeq 77.19^\circ$ and
$\widehat{C} \simeq 180^\circ - 70^\circ - 77.19^\circ \simeq 32.81^\circ$

Also allow the possibility that $\widehat{B} \simeq 102.81^\circ$ and $\widehat{C} \simeq 7.19^\circ$

However, alternative solution unacceptable since not consistent with all of the Sine Rule.

Thus,

$$a \simeq 8.67\text{cm}, \quad \widehat{B} \simeq 77.19^\circ, \quad \widehat{C} \simeq 32.81^\circ$$

Note: It is possible to encounter examples for which more than one solution **does** exist.