"JUST THE MATHS"

SLIDES NUMBER

2.4

SERIES 4
(Further convergence and divergence)

by

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UNIT 2.4 - SERIES 4 - FURTHER CONVERGENCE AND DIVERGENCE

2.4.1 SERIES OF POSITIVE AND NEGATIVE TERMS

TEST 1 - The r-th Term Test (Revisited)

The r-th Term Test in Unit 2.3 may used for series whose terms are not necessarily all positive.

Outline Proof:

The formula

$$u_r = S_r - S_{r-1}$$

is valid for any series.

The series cannot converge unless the partial sums S_r and S_{r-1} both tend to the same finite limit as r tends to infinity.

Hence, u_r tends to zero as r tends to infinity.

Alternating Series

A simple kind of series with both positive and negative terms is one whose terms are alternately positive and negative.

Test 4 - The Alternating Series Test

If

$$u_1 - u_2 + u_3 - u_4 + \ . \ . \ , \quad \text{where} \ \ u_r > 0,$$

is such that

$$u_r > u_{r+1} \ \ \textbf{and} \ \ u_r \to 0 \ \ \text{as} \ \ r \to \infty,$$

then the series converges.

Outline Proof:

(a) Re-group the series as

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ . \ . \ .;$$

That is,

$$\sum_{r=1}^{\infty} v_r$$

where $v_1 = u_1 - u_2, v_2 = u_3 - u_4, v_3 = u_5 - u_6, \ . \ . \ . \ .$

v_r is positive, so that

$$S_r = v_1 + v_2 + v_3 + \ . \ . \ . \ + v_r$$

increases as r increases.

(b) Alternatively, re-group the series as

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ . \ . \ . \ ;$$

That is,

$$u_1 - \sum_{r=1}^{\infty} w_r$$

where $w_1 = u_2 - u_3, w_2 = u_4 - u_5, w_3 = u_6 - u_7, \ . \ . \ . \ .$

$S_r = u_1 - (w_1 + w_2 + w_3 + \ . \ . \ . + w_r)$ is less than u_1 since positive quantities are being subtracted from it.

(c) We conclude that the partial sums of the original series are steadily increasing but are never greater than u_1.

They must therefore tend to a finite limit as r tends to infinity; that is, the series converges.

ILLUSTRATION

The series

$$\sum_{r=1}^{\infty} (-1)^{r-1}\frac{1}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots$$

is convergent since

$$\frac{1}{r} > \frac{1}{r+1} \ \textbf{and} \ \frac{1}{r} \to 0 \ \text{ as } \ r \to \infty.$$

2.4.2 ABSOLUTE AND CONDITIONAL CONVERGENCE

DEFINITION (A)

If

$$\sum_{r=1}^{\infty} u_r$$

is a series with both positive and negative terms, it is said to be **"absolutely convergent"** if

$$\sum_{r=1}^{\infty} |u_r|$$

is convergent.

DEFINITION (B)
If
$$\sum_{r=1}^{\infty} u_r$$
is a convergent series of positive and negative terms, but
$$\sum_{r=1}^{\infty} |u_r|$$
is a divergent series, then the first of these two series is said to be **"conditionally convergent"**.

ILLUSTRATIONS

1. The series
$$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$
converges absolutely since the series
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$
converges.
2. The series
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ \ldots$$
is conditionally convergent.
It converges (by the Alternating Series Test), but the series
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \ldots \ \text{diverges.}$$

Notes:

(i) It may be shown that any series of positive and negative terms which is **absolutely** convergent will also be convergent.

(ii) Any test for the convergence of a series of positive terms may be used as a test for the absolute convergence of a series of both positive and negative terms.

2.4.3 TESTS FOR ABSOLUTE CONVERGENCE

The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that $|u_r| \leq v_r$ where

$$\sum_{r=1}^{\infty} v_r$$

is a convergent series of positive terms. Then, the given series is absolutely convergent.

D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r \to \infty} \left| \frac{u_{r+1}}{u_r} \right| = L.$$

Then the given series is absolutely convergent if $L < 1$.

Note:

If $L > 1$, then $|u_{r+1}| > |u_r|$ for large enough values of r showing that the **numerical** values of the terms steadily increase.

This implies that u_r does **not** tend to zero as r tends to infinity

Hence, (by the r-th Term Test) the series diverges.

If $L = 1$, there is no conclusion.

EXAMPLES

1. Show that the series

$$\frac{1}{1\times 2}-\frac{1}{2\times 3}-\frac{1}{3\times 4}+\frac{1}{4\times 5}-\frac{1}{5\times 6}-\frac{1}{6\times 7}+\ \ldots$$

is absolutely convergent.

Solution

The r-th term of the series is numerically equal to

$$\frac{1}{r(r+1)}$$

This is always less than $\frac{1}{r^2}$, the r-th term of a known convergent series.

2. Show that the series

$$\frac{1}{2}-\frac{2}{5}+\frac{3}{10}-\frac{4}{17}+\ \ldots$$

is conditionally convergent.

Solution

The r-th term of the series is numerically equal to

$$\frac{r}{r^2+1},$$

which tends to zero as r tends to infinity.

Also,

$$\frac{r}{r^2+1} > \frac{r+1}{(r+1)^2+1}$$

since this may be reduced to the true statement $r^2 + r > 1$.

Hence, by the alternating series test, the series converges.

However,

$$\frac{r}{r^2+1} > \frac{r}{r^2+r} = \frac{1}{r+1}$$

and, hence, by the Comparison Test, the series of absolute values is divergent since

$$\sum_{r=1}^{\infty} \frac{1}{r+1}$$

is divergent.

2.4.4 POWER SERIES

A series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ .\ . \ = \sum_{r=0}^{\infty} a_r x^r \ \text{ or } \ \sum_{r=1}^{\infty} a_{r-1} x^{r-1},$$

where x is usually a variable quantity, is called a **"power series in x with coefficients,**
$a_0,\ a_1,\ a_2,\ a_3,\ .\ .\ .$".

Notes:

(i) By summing the series from $r = 0$ to infinity, the constant term at the beginning (if there is one) can be considered as the term in x^0.

The various tests for convergence and divergence still apply in this alternative notation.

(ii) A power series will not necessarily be convergent (or divergent) for **all** values of x

It is usually required to determine the specific **range** of values of x for which the series converges

This can most frequently be done using D'Alembert's Ratio Test

ILLUSTRATION

For the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ . \ . \ = \sum_{r=1}^{\infty} \ (-1)^{r-1}\frac{x^r}{r},$$

$$\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{(-1)^r x^{r+1}}{r+1}.\frac{r}{(-1)^{r-1}x^r}\right| = \left|\frac{r}{r+1}x\right|,$$

which tends to $|x|$ as r tends to infinity.

Thus, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

If $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ . \ .$$

which converges by the Alternating Series Test.

If $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ . \ .$$

which diverges.

The **precise** range of convergence for the given series is therefore $-1 < x \leq 1$.