“JUST THE MATHS”

SLIDES NUMBER

13.9

INTEGRATION APPLICATIONS 9
(First moments of a surface of revolution)

by

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UNIT 13.9 - INTEGRATION APPLICATIONS 9

FIRST MOMENTS OF A SURFACE OF REVOLUTION

13.9.1 INTRODUCTION

Let C denote an arc (with length s) in the xy-plane of cartesian co-ordinates.

Let δs denote the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"first moment"** about a plane through the origin, perpendicular to the x-axis, is given by

$$\lim_{\delta s \to 0} \sum_{\text{C}} 2\pi x y \delta s,$$

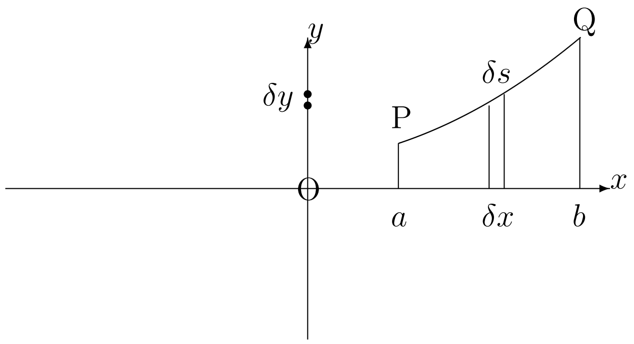
where x is the perpendicular distance, from the plane of moments, of the thin band, with surface area $2\pi y \delta s$, so generated.

13.9.2 INTEGRATION FORMULAE FOR FIRST MOMENTS

(a) Consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

For the surface of revolution of the arc about the x-axis, the first moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\delta x$$

$$= \int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t)$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative on the arc, then the previous line needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{t_1}^{t_2} 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

The first moment about the plane through the origin, perpendicular to the x-axis is given by

$$\text{First moment} = \pm\int_{t_1}^{t_2} 2\pi xy\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

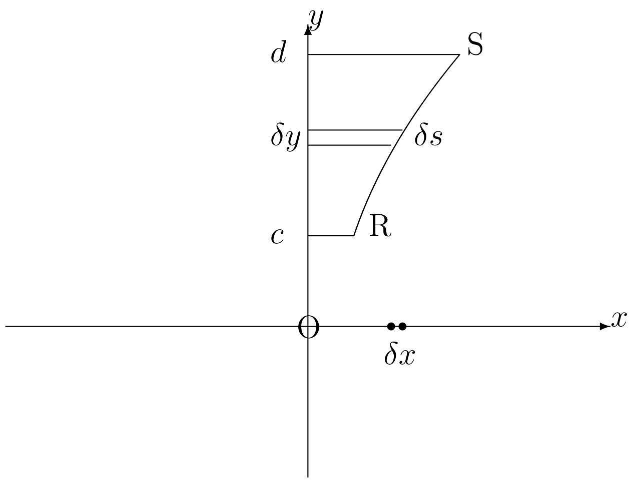
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the first moment about a plane through the origin, perpendicular to the y-axis is as follows:

$$\text{First moment} = \int_c^d 2\pi y x \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$



Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\text{First moment} = \pm \int_{t_1}^{t_2} 2\pi y x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

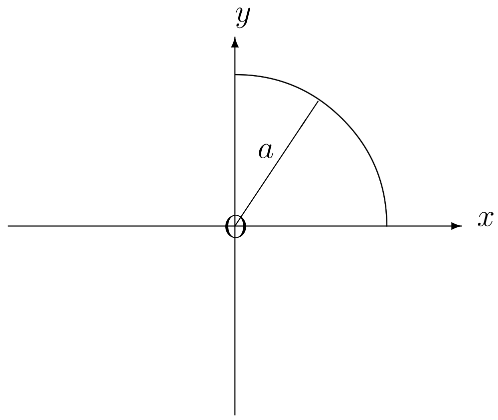
EXAMPLES

1. Determine the first moment about a plane through the origin, perpendicular to the x-axis, for the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant

Solution



$$2x + 2y\frac{dy}{dx} = 0.$$

Hence,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The first moment about the specified plane is therefore given by

$$\int_0^a 2\pi xy\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi xy\sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

$$\text{But}\ \ x^2+y^2=a^2.$$

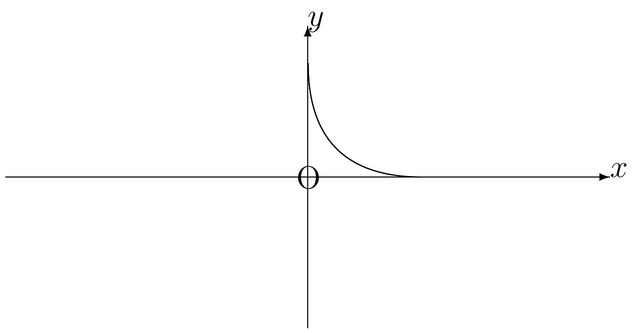
Thus, the first moment becomes

$$\int_0^a 2\pi ax\ \mathrm{d}x = [\pi ax^2]_0^a = \pi a^3.$$

2. Determine the first moments about planes through the origin, (a) perpendicular to the x-axis and (b) perpendicular to the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi xy\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

On using $\cos^2\theta + \sin^2\theta \equiv 1$, this becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^2\cos^3\theta\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} 6\pi a^3\cos^4\theta\sin^4\theta\ \mathrm{d}\theta.$$

Using $2\sin\theta\cos\theta \equiv \sin 2\theta$, the integral reduces to

$$\frac{3\pi a^3}{8}\int_0^{\frac{\pi}{2}} \sin^4 2\theta\ \mathrm{d}\theta$$

$$= \frac{3\pi a^3}{32}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right)\ \mathrm{d}\theta$$

$$= \frac{3\pi a^3}{32}\left[\frac{3\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{9\pi a^3}{128}.$$

By symmetry, or by direct integration, the first moment about a plane through the origin, perpendicular to the y-axis is also

$$\frac{9\pi a^3}{128}.$$

13.9.3 THE CENTROID OF A SURFACE OF REVOLUTION

Having calculated the first moment of a surface of revolution about a plane through the origin, perpendicular to the x-axis, it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $S\overline{x}$, where S is the total surface area.

The point is called the **"centroid"** or the **"geometric centre"** of the surface of revolution and, for the surface of revolution of the arc of the curve whose equation is $y = f(x)$, between $x = a$ and $x = b$, the value of $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b \ 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \ \mathrm{d}x}{\int_a^b \ 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}$$

$$= \frac{\int_a^b \ xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b \ y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \ \mathrm{d}x}.$$

Note: The centroid effectively tries to concentrate the whole surface at a single point for the purposes of considering first moments.

In practice, the centroid of a surface corresponds to the position of the centre of mass of a thin sheet, for example, with uniform density.

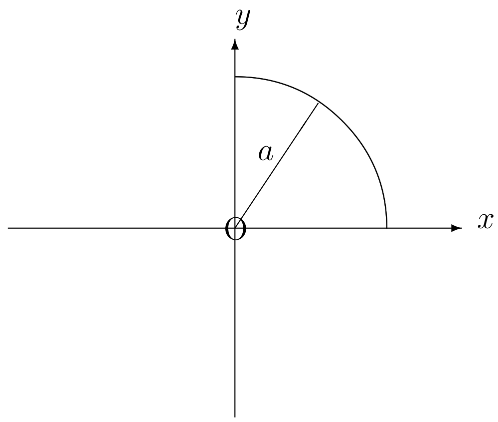
EXAMPLES

1. Determine the position of the centroid of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 in Section 13.9.2, the first moment of the surface about a plane through the origin, perpendicular to the the x-axis is equal to πa^3.

The total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\,\mathrm{d}x.$$

Using $x^2+y^2=a^2$,

$$\text{surface area} = \int_0^a 2\pi a\,\mathrm{d}x = 2\pi a^2.$$

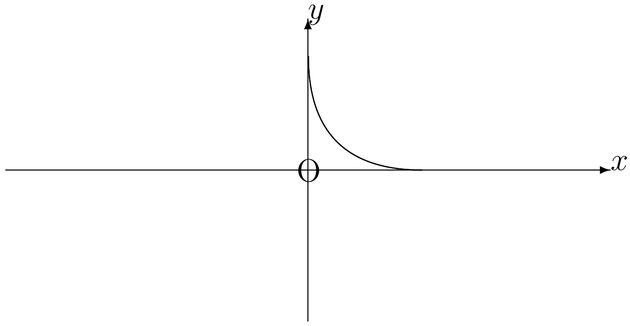
Hence,

$$\overline{x} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

2. Determine the position of the centroid of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.9.2, the first moment of the surface about a plane through the origin, perpendicular to the x-axis is equal to

$$\frac{9\pi a^3}{128}.$$

The total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a \sin^3\theta.3a\cos\theta\sin\theta \, \mathrm{d}\theta$$

$$= \int_{0}^{\frac{\pi}{2}} 3a^2 \sin^4\theta\cos\theta \, \mathrm{d}\theta$$

$$= 3\pi a^2 \left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$\overline{x} = \frac{9\pi a^3}{128} \div \frac{3\pi a^2}{5}$$

or

$$\overline{x} = \frac{15a}{128}.$$