"JUST THE MATHS"

SLIDES NUMBER

13.7

INTEGRATION APPLICATIONS 7
(First moments of an area)

by

A.J.Hobson

UNIT 13.7 - INTEGRATION APPLICATIONS 7

FIRST MOMENTS OF AN AREA

13.7.1 INTRODUCTION

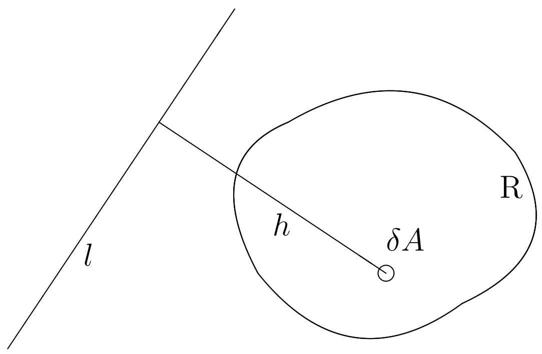
Let R denote a region (with area A) of the xy-plane of cartesian co-ordinates.

Let δA denote the area of a small element of this region.

Then, the **"first moment"** of R about a fixed line, l, in the plane of R is given by

$$\lim_{\delta A \to 0} \sum_{\text{R}} h\delta A,$$

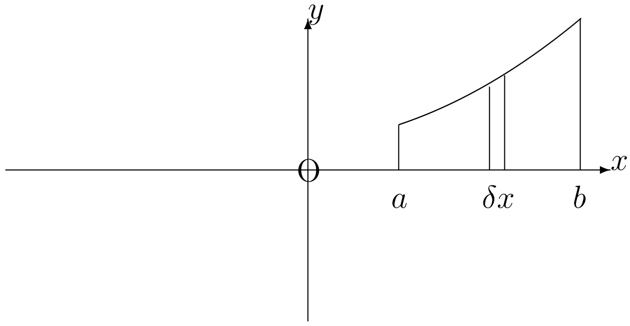
where h is the perpendicular distance, from l, of the element with area δA.



13.7.2 FIRST MOMENT OF AN AREA ABOUT THE Y-AXIS

Consider a region in the first quadrant of the xy-plane bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may divided up into small elements by using a network, of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

All of the elements in a narrow 'strip' of width δx and height y (parallel to the y-axis) have the same perpendicular distance, x, from the y-axis.

Hence the first moment of this strip about the y-axis is x $x(y\delta x)$.

Thus, the total first moment of the region about the y-axis is given by

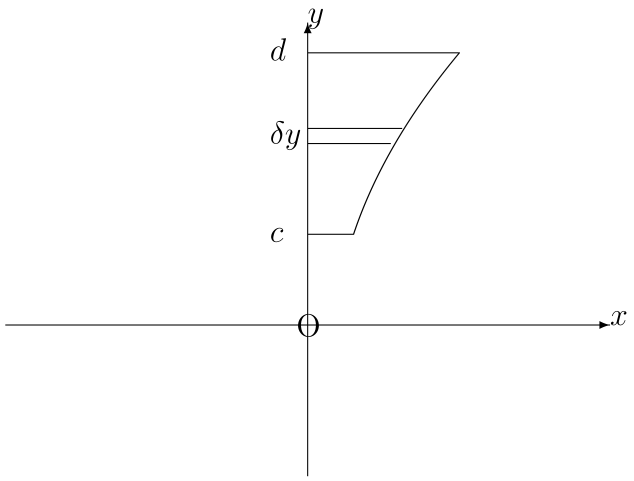
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy \, \mathrm{d}x.$$

Note:
For a region of the first quadrant bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment about the x-axis is given by

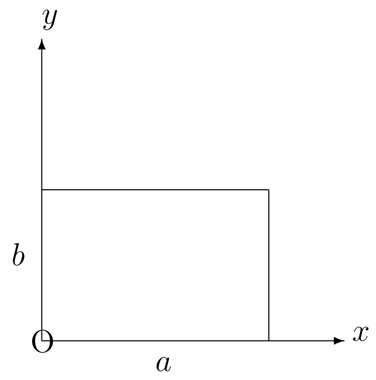
$$\int_c^d yx \, \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a rectangular region, with sides of lengths a and b, about the side of length b.

   Solution



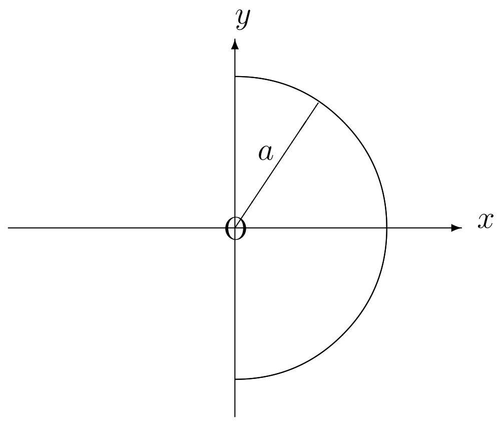
The first moment about the y-axis is given by

$$\int_0^a xb \, \mathrm{d}x = \left[\frac{x^2b}{2}\right]_0^a = \frac{1}{2}a^2b.$$

2. Determine the first moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the first moment about the y-axis is given by

$$2\int_0^a x\sqrt{a^2 - x^2}\,\mathrm{d}x = \left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}a^3.$$

Note:

The symmetry of the above region shows that its first moment about the x-axis would be zero.

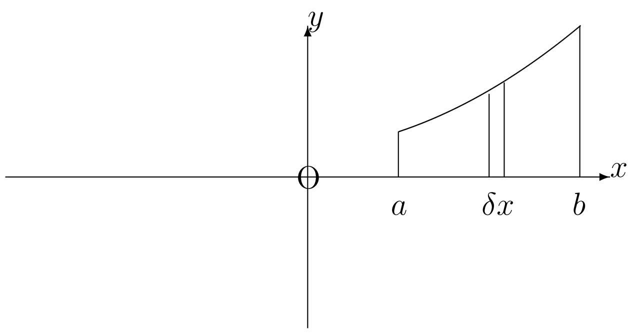
This is because, for each $y(x\delta y)$, there will be a corresponding $-y(x\delta y)$ in calculating the first moments of the strips parallel to the x-axis.

13.7.3 FIRST MOMENT OF AN AREA ABOUT THE X-AXIS

In Example 1 of Section 13.7.2, a formula was established for the first moment of a rectangular region about one of its sides.

This result may be used to determine the first moment about the x-axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow strip of width δx and height y is regarded as approximately a rectangle, its first moment about the x-axis is

$$\frac{1}{2}y^2\delta x.$$

Hence, the first moment of the whole region about the x-axis is given by

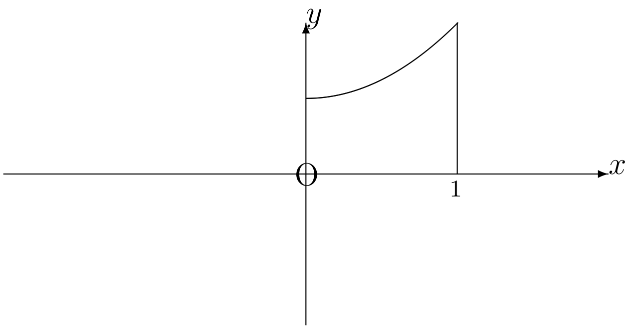
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{2} y^2 \delta x = \int_a^b \frac{1}{2} y^2 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the first moment about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



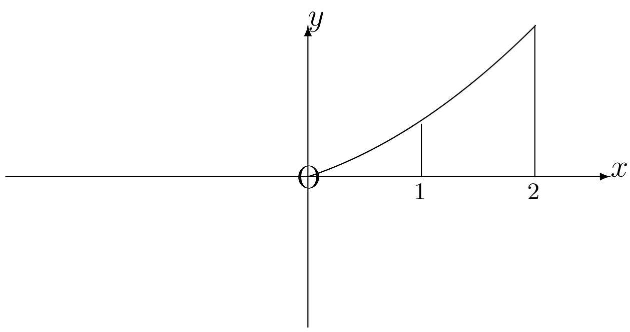
$$\text{First moment} = \int_0^1 \frac{1}{2}(x^2 + 1)^2 \, \mathrm{d}x$$

$$= \frac{1}{2} \int_0^1 (x^4 + 2x^2 + 1) \, \mathrm{d}x = \frac{1}{2} \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]_0^1 = \frac{28}{15}.$$

2. Determine the first moment about the x-axis of the region, bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



$$\text{First moment } = \int_1^2 \frac{1}{2}x^2 e^{2x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \int_1^2 xe^{2x}\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \left[x\frac{e^{2x}}{2}\right]_1^2 + \int_1^2 \frac{e^{2x}}{2}\,\mathrm{d}x\right).$$

That is,

$$\frac{1}{2}\left[x^2\frac{e^{2x}}{2} - x\frac{e^{2x}}{2} + \frac{e^{2x}}{4}\right]_1^2 = \frac{5e^4 - e^2}{8} \simeq 33.20$$

13.7.4 THE CENTROID OF AN AREA

Having calculated the first moments of a two dimensional region about both the x-axis and the y-axis, it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $A\overline{x}$, where A is the total area of the region

and

(b) The first moment about the x-axis is given by $A\overline{y}$, where A is the toal area of the region.

The point is called the **"centroid"** or the **"geometric centre"** of the region.

In the case of a region bounded in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b xy \, \mathrm{d}x}{\int_a^b y\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b \frac{1}{2}y^2 \, \mathrm{d}x}{\int_a^b y \, \mathrm{d}x}.$$

Notes:

(i) The first moment of an area about an axis through its centroid will, by definition, be zero.

In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δA, to the y-axis, the first moment about the given axis will be

$$\sum_{R} (x - \overline{x})\delta A = \sum_{R} x\delta A - \overline{x}\sum_{R} \delta A = A\overline{x} - A\overline{x} = 0.$$

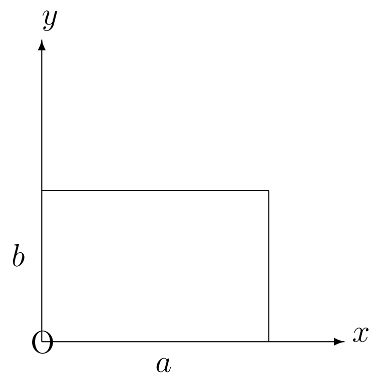
(ii) The centroid effectively tries to concentrate the whole area at a single point for the purposes of considering first moments.

In practice, the centroid corresponds to the position of the centre of mass for a thin plate with uniform density whose shape is that of the region considered.

EXAMPLES

1. Determine the position of the centroid of a rectangular region with sides of lengths, a and b.

Solution



The area of the rectangle is ab and the first moments about the y-axis and x-axis are

$$\frac{1}{2}a^2b \text{ and } \frac{1}{2}b^2a, \quad \text{respectively}$$

Hence,

$$\overline{x} = \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$

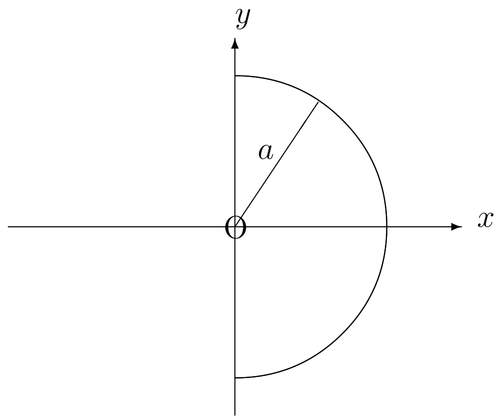
and

$$\overline{y} = \frac{\frac{1}{2}b^2a}{ab} = \frac{1}{2}b.$$

2. Determine the position of the centroid of the semi-circular region bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



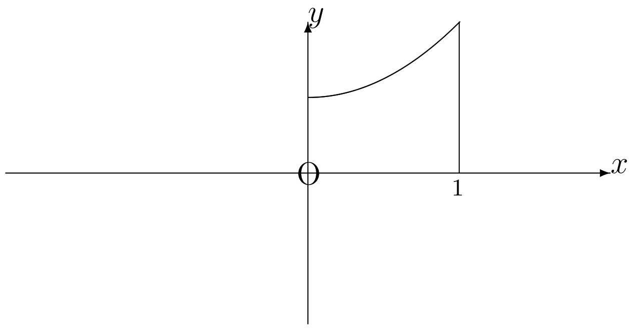
The area of the semi-circular region is $\frac{1}{2}\pi a^2$ and so, from Example 2 in section 13.7.2,

$$\overline{x} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \text{ and } \overline{y} = 0$$

3. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



The first moment about the y-axis is given by

$$\int_0^1 x(x^2+1)\,\mathrm{d}x = \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^1 = \frac{3}{4}.$$

The area is given by

$$\int_0^1 (x^2+1)\,\mathrm{d}x = \left[\frac{x^3}{3}+x\right]_0^1 = \frac{4}{3}.$$

Hence,

$$\overline{x} = \frac{3}{4} \div \frac{4}{3} = 1.$$

The first moment about the x-axis is $\frac{28}{15}$ from Example 1 in Section 13.7.3.

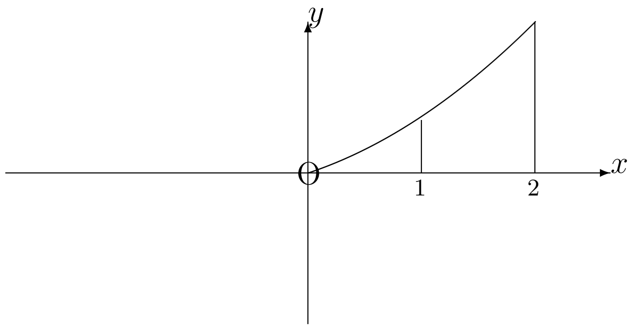
Therefore,

$$\overline{y} = \frac{28}{15} \div \frac{4}{3} = \frac{7}{5}.$$

4. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = xe^x.$$

Solution



The first moment about the y-axis is given by

$$\int_1^2 x^2 e^x \, \mathrm{d}x = \left[x^2 e^x - 2xe^x + 2e^x\right]_1^2 \simeq 12.06$$

$$\text{The area} = \int_1^2 xe^x \, \mathrm{d}x = \left[xe^x - e^x\right]_1^2 \simeq 7.39$$

$$\text{Hence } \overline{x} \simeq 12.06 \div 7.39 \simeq 1.63$$

The first moment about the x-axis is approximately 33.20, from Example 2 in Section 13.7.3.

$$\text{Thus } \overline{y} \simeq 33.20 \div 7.39 \simeq 4.47$$