"JUST THE MATHS"

SLIDES NUMBER

13.6

INTEGRATION APPLICATIONS 6
(First moments of an arc)

by

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UNIT 13.6 - INTEGRATION APPLICATIONS 6

FIRST MOMENTS OF AN ARC

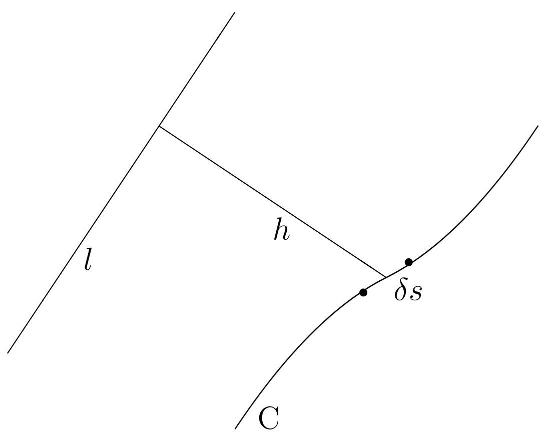
13.6.1 INTRODUCTION

Let C denote an arc (with length s) in the xy-plane of cartesian co-ordinates, and let δs be the length of a small element of this arc.

Then, the **"first moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h\delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

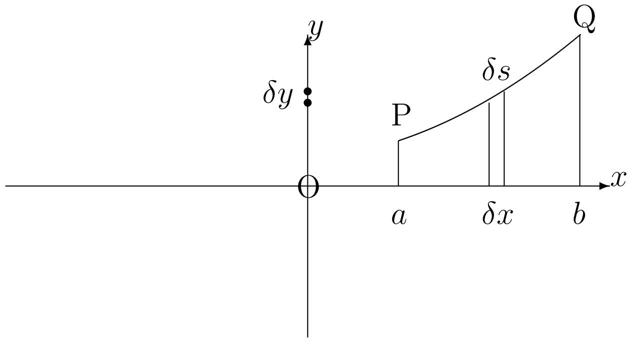


13.6.2 FIRST MOMENT OF AN ARC ABOUT THE Y-AXIS

Consider an arc of the curve, with equation

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The first moment of each element about the y-axis is $x\delta s$.

Hence, the total first moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x\delta s.$$

But, by Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x.$$

Thus, the first moment of the arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x$$

$$= \int_a^b x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered.

If $\frac{\mathrm{d}x}{\mathrm{d}t}$ is negative on the arc, then the above formula needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{t_1}^{t_2} x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

Thus, the first moment of the arc about the y-axis is given by

$$\pm\int_{t_1}^{t_2} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.6.3 FIRST MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the first moment about the x-axis will be

$$\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

the first moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

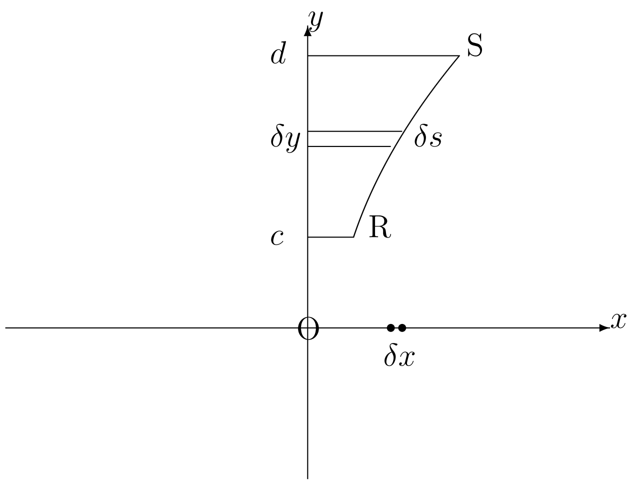
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.6.2 so that the first moment about the x-axis is given by

$$\int_c^d y\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then the first moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt,$$

according as $\frac{dy}{dt}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

EXAMPLES

1. Determine the first moments about the x-axis and the y-axis of the arc of the circle, with equation

   $$x^2 + y^2 = a^2,$$

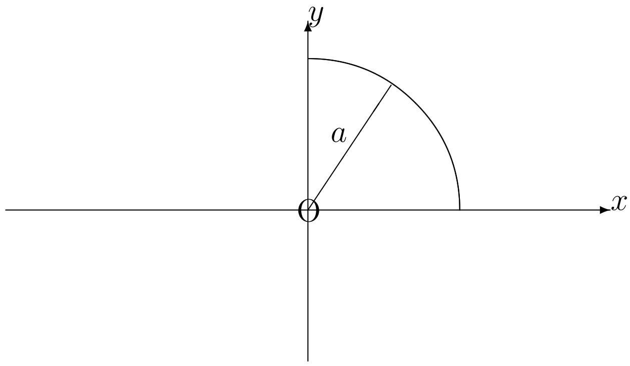
   lying in the first quadrant.

   Solution

   Using implicit differentiation

   $$2x + 2y\frac{dy}{dx} = 0.$$

$$\text{Hence,} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$



The first moment about the y-axis is given by

$$\int_0^a x\sqrt{1+\frac{x^2}{y^2}}\,\mathrm{d}x = \int_0^a \frac{x}{y}\sqrt{x^2+y^2}\,\mathrm{d}x.$$

But

$$x^2+y^2=a^2 \ \text{ and } \ y=\sqrt{a^2-x^2}.$$

Hence,

$$\text{first moment} = \int_0^a \frac{ax}{\sqrt{a^2-x^2}}\,\mathrm{d}x$$

$$= \left[-a\sqrt{(a^2-x^2)}\right]_0^a = a^2.$$

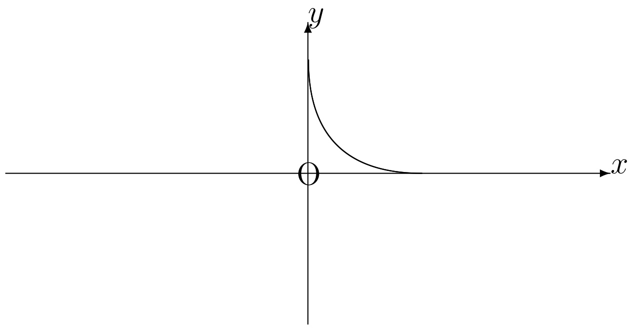
By symmetry, the first moment about the x-axis will also be a^2.

2. Determine the first moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$



The first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} y\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

Using $\cos^2\theta + \sin^2\theta \equiv 1$, this becomes

$$\int_0^{\frac{\pi}{2}} a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}.$$

Similarly, the first moment about the y-axis is given by

$$\int_0^{\frac{\pi}{2}} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} a\cos^3\theta.(3a\cos\theta\sin\theta)\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[-\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3a^2}{5}.$$

Note:
This second result could be deduced, by symmetry, from the first.

13.6.4 THE CENTROID OF AN ARC

Having calculated the first moments of an arc about both the x-axis and the y-axis it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $s\overline{x}$, where s is the total length of the arc;

and

(b) The first moment about the x-axis is given by $s\overline{y}$, where s is the total length of the arc.

The point is called the **"centroid"** or the **"geometric centre"** of the arc.

For an arc of the curve, with equation $y = f(x)$, between $x = a$ and $x = b$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}.$$

Notes:

(i) The first moment of an arc about an axis through its centroid will, by definition, be zero.

In particular, let the y-axis be parallel to the given axis.

Let x be the perpendicular distance from an element, δs, to the y-axis.

The first moment about the given axis will be

$$\sum_{\mathrm{C}} (x - \overline{x})\delta s = \sum_{\mathrm{C}} x\delta s - \overline{x}\sum_{\mathrm{C}} \delta s = s\overline{x} - s\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole arc at a single point for the purposes of considering first moments.

In practice, the centroid corresponds, for example, to the position of the centre of mass of a thin wire with uniform density.

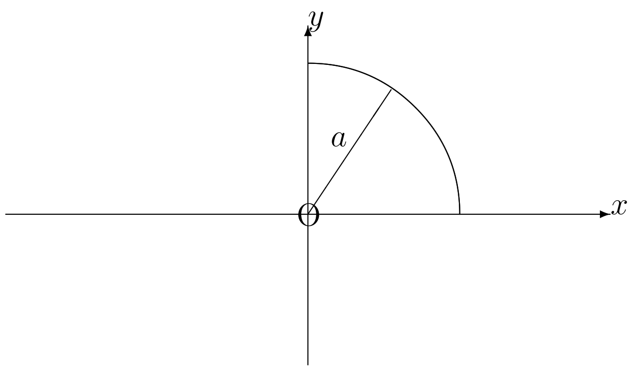
EXAMPLES

1. Determine the cartesian co-ordinates of the centroid of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant

Solution



From an earlier example in this unit, the first moments of the arc about the x-axis and the y-axis are both equal to a^2.

Also, the length of the arc is $\frac{\pi a}{2}$.

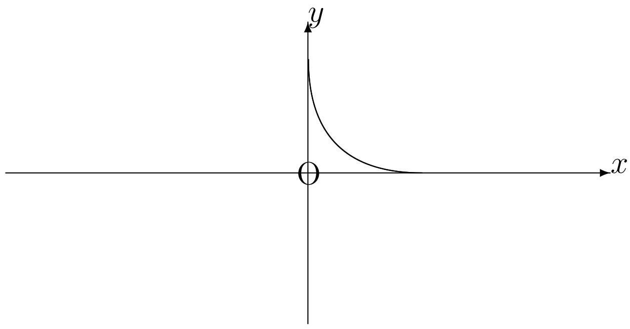
Hence,

$$\overline{x} = \frac{2a}{\pi} \text{ and } \overline{y} = \frac{2a}{\pi}.$$

2. Determine the cartesian co-ordinates of the centroid of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From an earlier example in this unit,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

The first moments of the arc about the x-axis and the y-axis are both equal to $\frac{3a^2}{5}$.

Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus,

$$\overline{x} = \frac{2a}{5} \ \text{ and } \ \overline{y} = \frac{2a}{5}.$$