

“JUST THE MATHS”

SLIDES NUMBER

13.5

INTEGRATION APPLICATIONS 5
(Surfaces of revolution)

by

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13.5.1 Surfaces of revolution about the x -axis
13.5.2 Surfaces of revolution about the y -axis

UNIT 13.5 - INTEGRATION APPLICATIONS 5

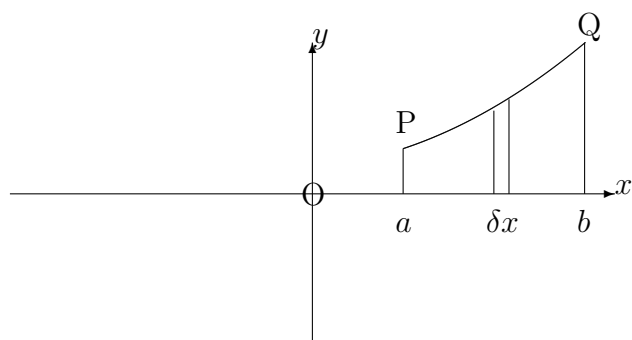
SURFACES OF REVOLUTION

13.5.1 SURFACES OF REVOLUTION ABOUT THE X-AXIS

The problem is to calculate the surface area obtained when the arc of the curve, with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$ respectively, is rotated through 2π radians about the x -axis or the y -axis.



For two neighbouring points along the curve, the arc joining them may be considered, approximately as a straight line segment.

If the two neighbouring points are separated by distances of δx and δy , parallel to the x -axis and the y -axis respectively, then the length, δs , of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x,$$

using Pythagoras's Theorem.

When the arc, of length δs is rotated through 2π radians about the x -axis, it generates a thin band whose area is, approximately,

$$2\pi y \delta s = 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

The total surface area, S , is thus given by

$$S = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

That is,

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Hence,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\frac{dx}{dt}},$$

provided $\frac{dx}{dt}$ is positive on the arc being considered.

If $\frac{dx}{dt}$ is negative on the arc, then the above result needs to be prefixed by a negative sign.

Using integration by substitution,

$$\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{t_1}^{t_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{dx}{dt} dt,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$S = \pm \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as $\frac{dx}{dt}$ is positive or negative.

EXAMPLES

1. A curve has equation

$$y^2 = 2x.$$

Determine the surface area obtained when the arc of the curve between the point $(2, 2)$ and the point $(8, 4)$ is rotated through 2π radians about the x -axis.

Solution

The equation of the arc may be written

$$y = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}};$$

and so,

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2x}}.$$

Hence,

$$S = \int_2^8 2\pi\sqrt{2x} \sqrt{1 + \frac{1}{2x}} dx$$

$$\begin{aligned}
&= \int_2^8 \sqrt{2x+1} \, dx \\
&= \left[\frac{(2x+1)^{\frac{3}{2}}}{3} \right]_2^8.
\end{aligned}$$

Thus,

$$S = \frac{17^{\frac{3}{2}}}{3} - \frac{5^{\frac{3}{2}}}{3} \simeq 19.64$$

2. A curve is given parametrically by

$$x = \sqrt{2} \cos \theta, \quad y = \sqrt{2} \sin \theta.$$

Determine the surface area obtained when the arc of the curve between the point $(0, \sqrt{2})$ and the point $(1, 1)$ is rotated through 2π radians about the x -axis.

Solution

The parameters of the two points are $\frac{\pi}{2}$ and $\frac{\pi}{4}$, respectively.

Also,

$$\frac{dx}{d\theta} = -\sqrt{2} \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = \sqrt{2} \cos \theta.$$

Thus,

$$S = - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\sqrt{2}\pi \sin \theta \sqrt{2\sin^2 \theta + 2\cos^2 \theta} \, d\theta$$

$$= - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\pi \sin \theta \, d\theta.$$

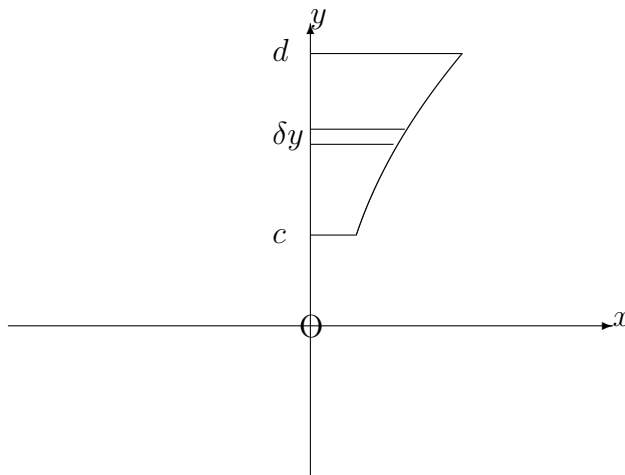
That is,

$$S = - [-4\pi \cos \theta]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{4\pi}{\sqrt{2}} \simeq 8.89$$

13.5.2 SURFACES OF REVOLUTION ABOUT THE Y-AXIS

For a curve whose equation is of the form $x = g(y)$, the surface of revolution about the y -axis of an arc joining the two points at which $y = c$ and $y = d$ is given by

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.$$



We simply reverse the previous roles of x and y .

If the curve is given parametrically,

$$S = \pm \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as $\frac{dy}{dt}$ is positive or negative.

EXAMPLE

If the arc of the parabola, with equation

$$x^2 = 2y,$$

joining the two points $(2, 2)$ and $(4, 8)$, is rotated through 2π radians about the y -axis, determine the surface area obtained.

Solution

Using a result from the previous section, the surface area obtained is given by

$$S = \int_2^8 2\pi\sqrt{2y} \sqrt{1 + \frac{1}{2y}} dy \simeq 19.64$$