"JUST THE MATHS"

SLIDES NUMBER

13.3

INTEGRATION APPLICATIONS 3
(Volumes of revolution)

by

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UNIT 13.3
INTEGRATION APLICATIONS 3

VOLUMES OF REVOLUTION

13.3.1 VOLUMES OF REVOLUTION ABOUT THE X-AXIS

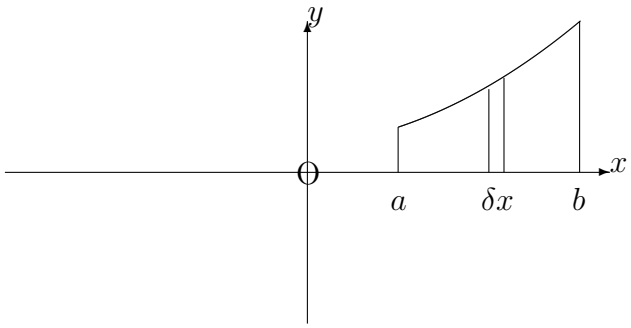
Suppose that the area between a curve

$$y = f(x)$$

and the x-axis, from $x = a$ to $x = b$ lies wholly above the x-axis.

Let this area be rotated through 2π radians about the x-axis.

Then, a solid figure is obtained whose volume may be determined as an application of definite integration.



Rotating a narrow strip of width δx and height y through 2π radians about the x-axis gives a disc.

The volume, δV, of the disc is given approximately by

$$\delta V \simeq \pi y^2 \delta x.$$

Thus, the total volume, V, is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x.$$

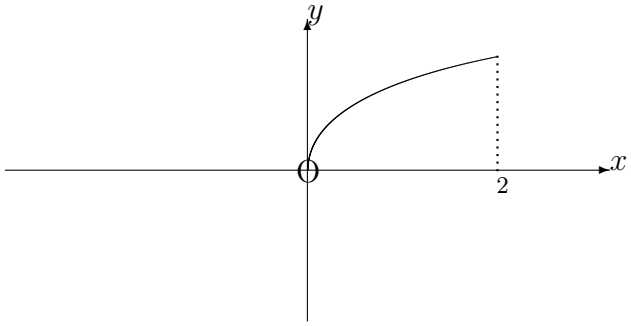
That is,

$$V = \int_a^b \pi y^2 \, \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the x-axis.

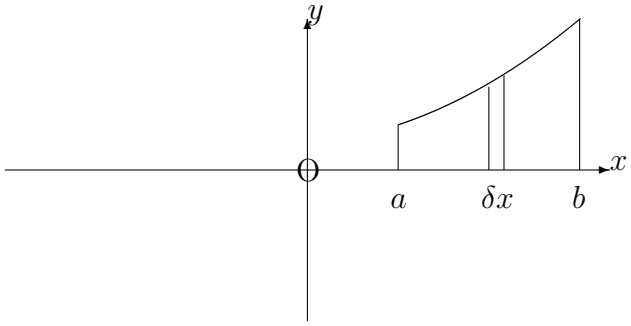
Solution



$$V = \int_0^2 \pi \times 8x \, \mathrm{d}x = [4\pi x^2]_0^2 = 16\pi.$$

13.3.2 VOLUMES OF REVOLUTION ABOUT THE Y-AXIS

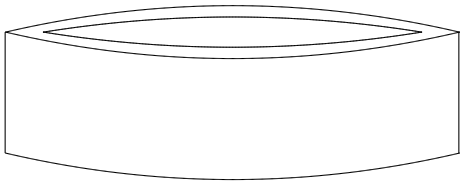
Consider the previous diagram:



Rotating the narrow strip of width δx through 2π radians about the y-axis gives a cylindrical shell of internal radius, x, external radius, $x + \delta x$ and height, y.

The volume, δV, of the shell is given by

$$\delta V \simeq 2\pi xy\delta x.$$



The total volume is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi xy\delta x.$$

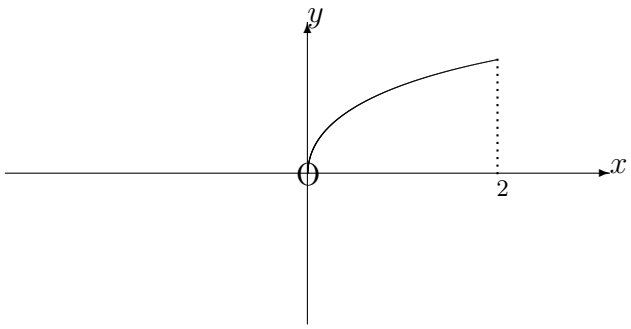
That is,

$$V = \int_a^b 2\pi xy \, \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the y-axis.

Solution



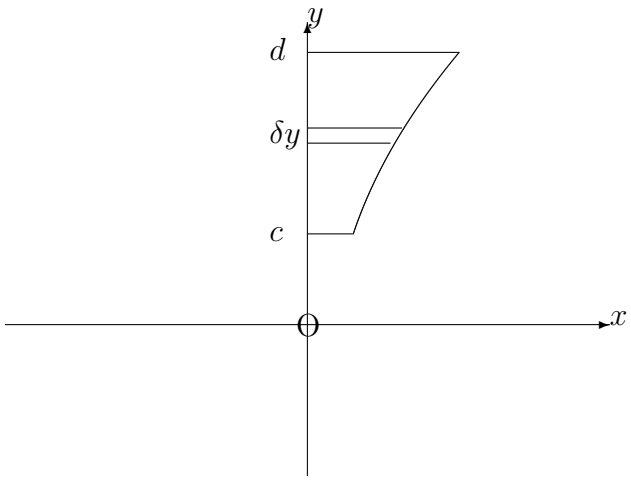
$$V = \int_0^2 \, 2\pi x \times \sqrt{8x} \, \mathrm{d}x.$$

In other words,

$$V = \pi 4\sqrt{2} \int_0^2 \, x^{\frac{3}{2}} \mathrm{d}x = \pi 4\sqrt{2} \left[\frac{2x^{\frac{5}{2}}}{5} \right]_0^2 = \frac{64\pi}{5}.$$

Note:

It may be required to find the volume of revolution about the y-axis of an area which is contained between a curve and the y-axis from $y = c$ to $y = d$.



Here, we interchange the roles of x and y in the original formula for rotation about the x-axis.

$$V = \int_c^d \pi x^2 \, \mathrm{d}y.$$

Similarly, the volume of rotation of the above area about the x-axis is given by

$$V = \int_c^d 2\pi yx \, \mathrm{d}y.$$