"JUST THE MATHS"

SLIDES NUMBER

13.13

INTEGRATION APPLICATIONS 13 (Second moments of a volume (A))

by

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UNIT 13.13 - INTEGRATION APPLICATIONS 13

SECOND MOMENTS OF A VOLUME (A)

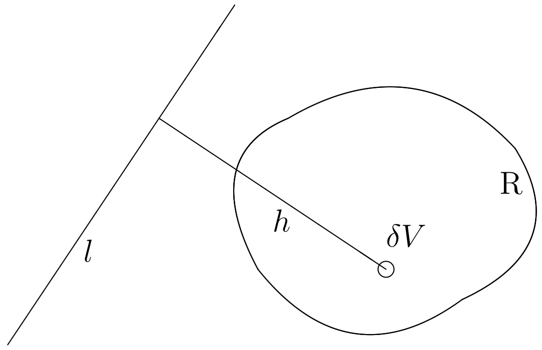
13.13.1 INTRODUCTION

Let R denote a region (with volume V) in space and suppose that δV is the volume of a small element of this region

Then the **"second moment"** of R about a fixed line, l, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h^2 \delta V,$$

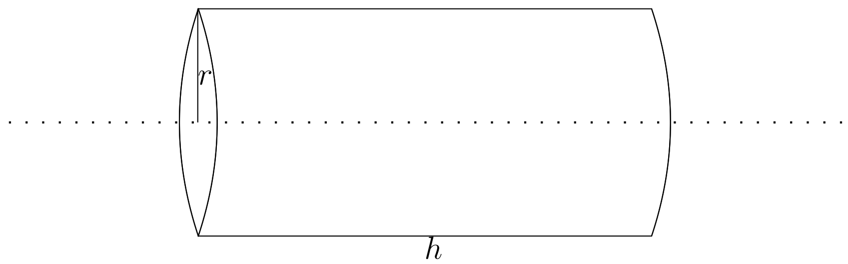
where h is the perpendicular distance from l of the element with volume, δV.



EXAMPLE

Determine the second moment, about its own axis, of a solid right-circular cylinder with height, h, and radius, a.

Solution



In a thin cylindrical shell with internal radius, r, and thickness, δr, all of the elements of volume have the same perpendicular distance, r, from the axis of moments.

Hence the second moment of this shell is

$$r^2(2\pi r h \delta r).$$

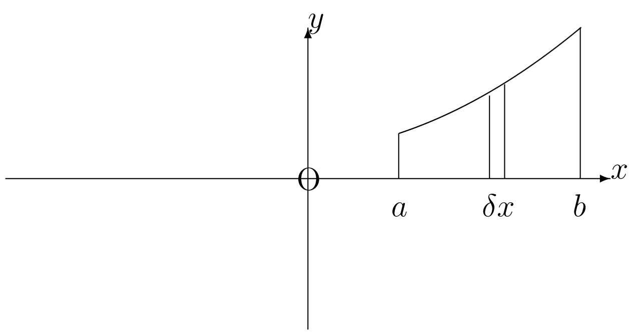
The total second moment is therefore given by

$$\lim_{\delta r \to 0} \sum_{r=0}^{r=a} r^2(2\pi r h \delta r) = \int_0^a 2\pi h r^3 \, \mathrm{d}r = \frac{\pi a^4 h}{2}.$$

13.13.2 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE Y-AXIS

Consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution of a narrow 'strip', of width δx, and height, y, (parallel to the y-axis), is a cylindrical 'shell', of internal radius x, height, y, and thickness, δx.

Hence, from the example in the previous section, its second moment about the y-axis is

$$2\pi x^3 y \delta x.$$

Thus, the total second moment about the y-axis is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x^3 y \delta x$$

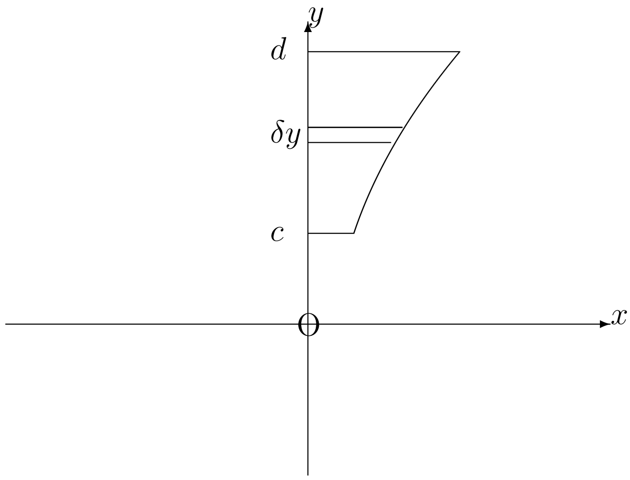
$$= \int_a^b 2\pi x^3 y \, \mathrm{d}x.$$

Note:
For the volume of revolution, about the x-axis, of a region in the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

$$\int_c^d 2\pi y^3 x \, \mathrm{d}y.$$



EXAMPLE

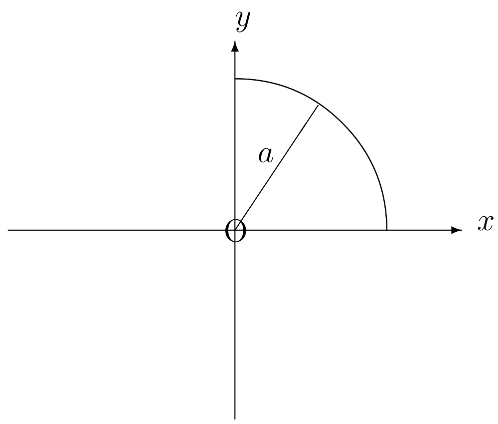
Determine the second moment, about a diameter, of a solid sphere with radius a.

Solution

We may consider, first, the volume of revolution about the y-axis of the region bounded in the first quadrant by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2,$$

then double the result obtained.



The total second moment is given by

$$2\int_0^a \ 2\pi x^3\sqrt{a^2-x^2}\ \mathrm{d}x$$

$$=4\pi\int_0^{\frac{\pi}{2}} \ a^3\sin^3\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x=a\sin\theta$.

This simplifies to

$$4\pi a^5\int_0^{\frac{\pi}{2}} \ \sin^3\theta\ \cos^2\theta\mathrm{d}\theta$$

$$=4\pi\int_0^{\frac{\pi}{2}} \ \left(\cos^2\theta-\cos^4\theta\right)\sin\theta\mathrm{d}\theta,$$

if we make use of the trigonometric identity

$$\sin^2\theta\equiv 1-\cos^2\theta.$$

The total second moment is now given by

$$4\pi a^5\left[-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}}$$

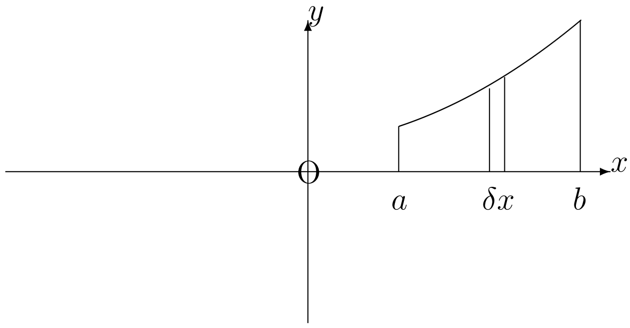
$$= 4\pi a^5\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{8\pi a^5}{15}.$$

13.13.3 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE X-AXIS

In the introduction to this Unit, a formula was established for the second moment of a solid right-circular cylinder about its own axis.

This result may now be used to determine the second moment, about the x-axis, for the volume of revolution about this axis, of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution about the x-axis of a narrow strip, of width δx and height y, is a cylindrical 'disc' whose second moment about the x-axis is

$$\frac{\pi y^4 \delta x}{2}.$$

Hence, the second moment of the whole region about the x-axis is given by

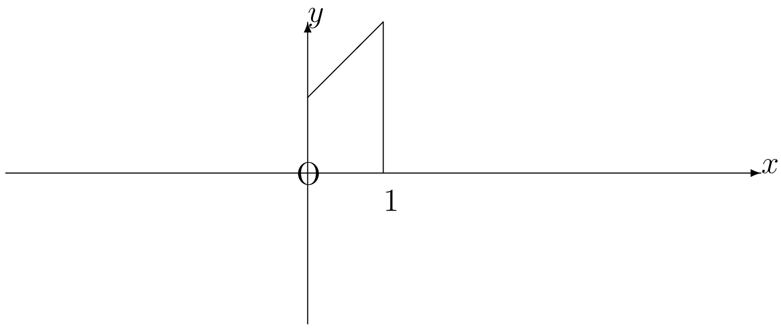
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{\pi y^4}{2} \delta x = \int_a^b \frac{\pi y^4}{2} \, \mathrm{d}x.$$

EXAMPLE

Determine the second moment about the x-axis, for the volume of revolution about this axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{\pi(x+1)^4}{2}\, \mathrm{d}x$$

$$= \left[\pi\frac{(x+1)^4}{10}\right]_0^1 = \frac{31\pi}{10}.$$

Note:
The second moment of a volume about a certain axis is closely related to its **"moment of inertia"** about that axis

In fact, for a solid with uniform density, ρ, the moment of inertia is ρ times the second moment of volume, since multiplication by ρ, of elements of volume, converts them into elements of mass