"JUST THE MATHS"

SLIDES NUMBER

13.12

INTEGRATION APPLICATIONS 12
(Second moments of an area (B))

by

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13.12.1 The parallel axis theorem
13.12.2 The perpendicular axis theorem
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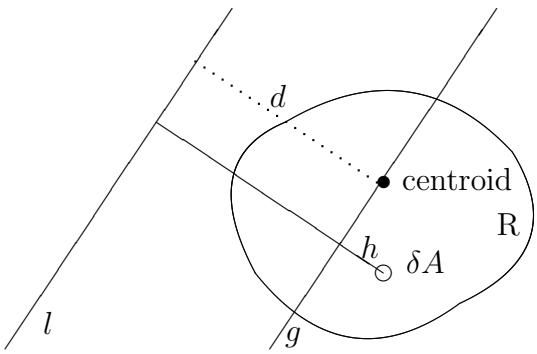
UNIT 13.12 - INTEGRATION APPLICATIONS 12

SECOND MOMENTS OF AN AREA (B)

13.12.1 THE PARALLEL AXIS THEOREM

Let M_g denote the second moment of a given region, R, about an axis, g, through its centroid.

Let M_l denote the second moment of R about an axis, l, which is parallel to the first axis, in the same plane as R and having a perpendicular distance of d from the first axis.



We have

$$M_l = \sum_{\mathrm{R}} (h+d)^2 \delta A = \sum_{\mathrm{R}} (h^2 + 2hd + d^2)$$

That is,

$$M_l = \sum_{\mathrm{R}} h^2 \delta A + 2d \sum_{\mathrm{R}} h \delta A + d^2 \sum_{\mathrm{R}} \delta A = M_g + Ad^2$$

Note:
The summation, $\Sigma_{\mathrm{R}}\ h\delta A$, is the first moment about the an axis through the centroid and, therefore, zero.

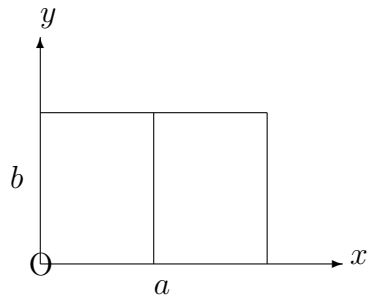
The Parallel Axis Theorem states that

$$M_l = M_g + Ad^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region about an axis through its centroid, parallel to one side.

   Solution



For a rectangular region with sides of length a and b, the second moment about the side of length b is

$$\frac{a^3 b}{3}.$$

The perpendicular distance between the two axes is $\frac{a}{2}$. Hence,

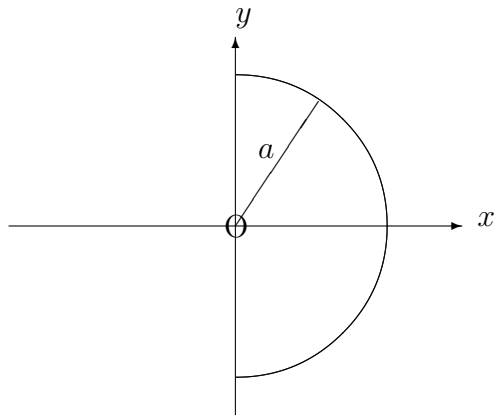
$$\frac{a^3 b}{3} = M_g + ab\left(\frac{a}{2}\right)^2 = M_g + \frac{a^3 b}{4},$$

giving

$$M_g = \frac{a^3 b}{12}.$$

2. Determine the second moment of a semi-circular region about an axis through its centroid, parallel to its diameter.

Solution



The second moment of the semi-circular region about its diameter is

$$\frac{\pi a^4}{8}.$$

The position of the centroid is a distance of

$$\frac{4a}{3\pi}$$

from the diameter along the radius which perpendicular to it.

Hence,

$$\frac{\pi a^4}{8} = M_g + \frac{\pi a^2}{2}.\left(\frac{4a}{3\pi}\right)^2 = M_g + \frac{8a^4}{9\pi^2}$$

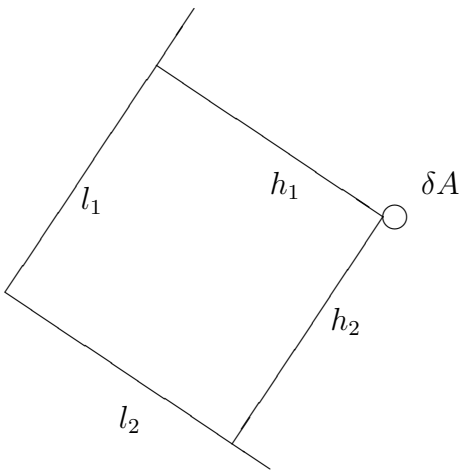
That is,

$$M_g = \frac{\pi a^4}{8} - \frac{8a^4}{9\pi^2}.$$

13.12.2 THE PERPENDICULAR AXIS THEOREM

Let l_1 and l_2 denote two straight lines, at right-angles to each other, in the plane of a region R with area A.

Let h_1 and h_2 be the perpendicular distances from these two lines respectively of an element δA in R.



The second moment about l_1 is given by

$$M_1 = \sum_R h_1^2 \delta A.$$

The second moment about l_2 is given by

$$M_2 = \sum_R h_2^2 \delta A.$$

Adding these two together gives the second moment about an axis perpendicular to the plane of R and passing through the point of intersection of l_1 and l_2.

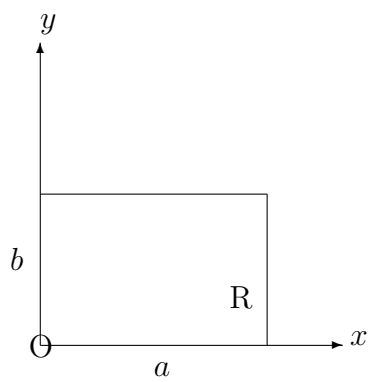
This is because the square of the perpendicular distance, h_3 of δA from this new axis is given, from Pythagoras's Theorem, by

$$h_3^2 = h_1^2 + h_2^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region, R, with sides of length a and b about an axis through one corner, perpendicular to the plane of R.

   Solution

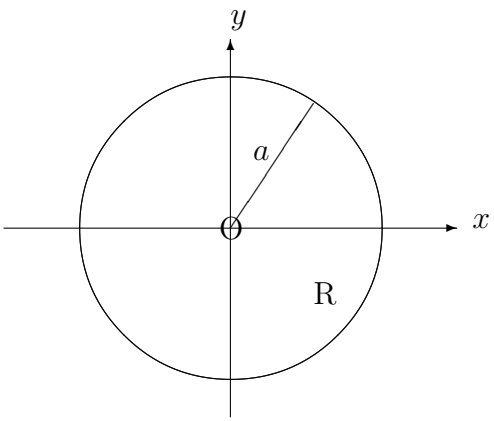


   The required second moment is

$$\frac{1}{3}a^3b + \frac{1}{3}b^3a = \frac{1}{3}ab(a^2 + b^2).$$

2. Determine the second moment of a circular region, R, with radius a, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment of R about a diameter is

$$\frac{\pi a^4}{4}.$$

That is, twice the value of the second moment of a semi-circular region about its diameter.

The required second moment is thus

$$\frac{\pi a^4}{4} + \frac{\pi a^4}{4} = \frac{\pi a^4}{2}.$$

13.12.3 THE RADIUS OF GYRATION OF AN AREA

Having calculated the second moment of a two-dimensional region about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Ak^2, where A is the total area of the region.

We divide the value of the second moment by A in order to obtain the value of k^2 and hence the value of k.

The value of k is called the **"radius of gyration"** of the given region about the given axis.

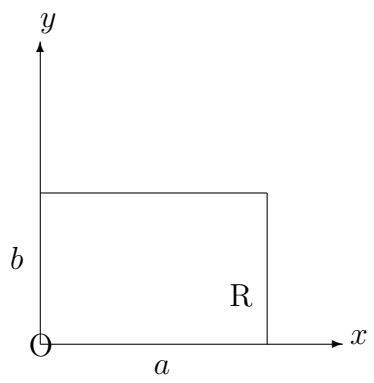
Note:
The radius of gyration effectively tries to concentrate the whole area at a single point for the purposes of considering second moments.

Unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a rectangular region, R, with sides of lengths a and b about an axis through one corner, perpendicular to the plane of R.

Solution



The second moment is

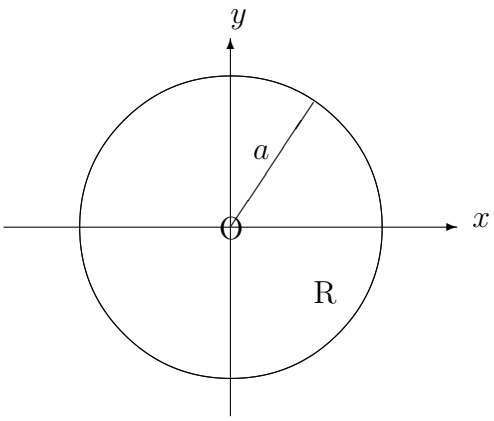
$$\frac{1}{3}ab(a^2 + b^2).$$

Since the area itself is ab, we obtain

$$k = \sqrt{a^2 + b^2}.$$

2. Determine the radius of gyration of a circular region, R, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment about the given axis is

$$\frac{\pi a^4}{2}.$$

Since the area itself is πa^2, we obtain

$$k = \frac{a}{\sqrt{2}}.$$