"JUST THE MATHS"

SLIDES NUMBER

13.11

INTEGRATION APPLICATIONS 11
(Second moments of an area (A))

by

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UNIT 13.11 - INTEGRATION APPLICATIONS 11

SECOND MOMENTS OF AN AREA (A)

13.11.1 INTRODUCTION

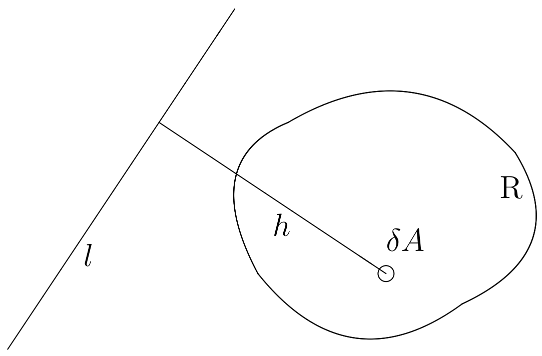
Let R denote a region (with area A) of the xy-plane in cartesian co-ordinates.

Let δA denote the area of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, **not necessarily in the plane of** R, is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h^2 \delta A,$$

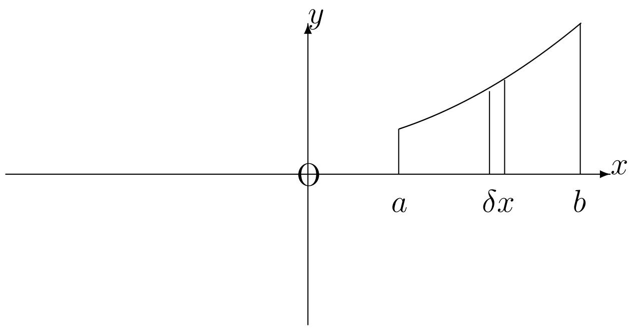
where h is the perpendicular distance from l of the element with area, δA.



13.11.2 THE SECOND MOMENT OF AN AREA ABOUT THE Y-AXIS

Consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may be divided up into small elements by using a network consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

All of the elements in a narrow 'strip', of width δx and height y (parallel to the y-axis), have the same perpendicular distance, x, from the y-axis.

Hence, the second moment of this strip about the y-axis is $x^2(y\delta x)$.

The total second moment of the region about the y-axis is given by

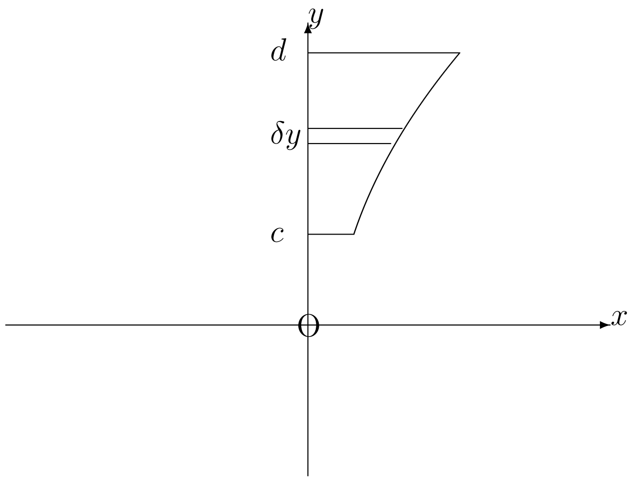
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 y \delta x = \int_a^b x^2 y \, \mathrm{d}x.$$

Note:
For a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

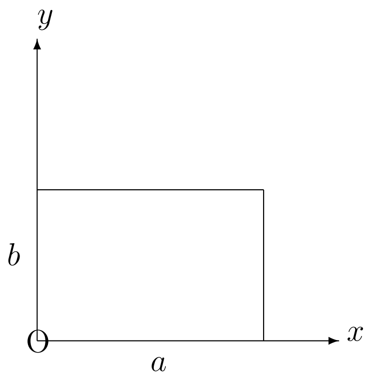
$$\int_c^d y^2 x \, \mathrm{d}y.$$



EXAMPLES

1. Determine the second moment of a rectangular region with sides of lengths, a and b, about the side of length b.

   Solution



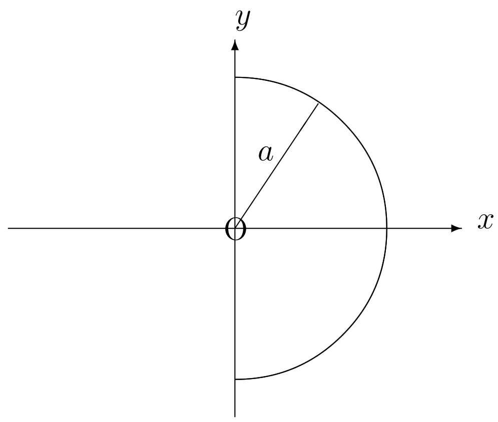
The second moment about the y-axis is given by

$$\int_0^a x^2 b \, \mathrm{d}x = \left[\frac{x^3 b}{3}\right]_0^a = \frac{1}{3}a^3 b.$$

2. Determine the second moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



There will be equal contributions from the upper and lower halves of the region.

Hence, the second moment about the y-axis is given by

$$2\int_0^a x^2\sqrt{a^2 - x^2}\ \mathrm{d}x$$

$$= 2\int_0^{\frac{\pi}{2}} a^2\sin^2\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

This simplifies to

$$2a^4 \int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4} \, \mathrm{d}\theta$$

$$= \frac{a^4}{2} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2} \, \mathrm{d}\theta$$

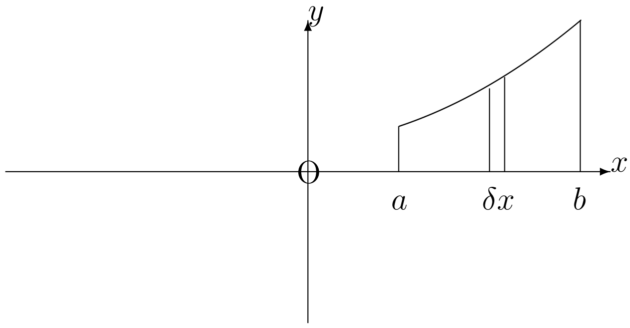
$$= \frac{a^4}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}.$$

13.11.3 THE SECOND MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the second moment of a rectangular region about one of its sides.

This result may now be used to determine the second moment about the x-axis of a region, enclosed in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow 'strip', of width δx and height y, is regarded, approximately, as a rectangle, its second moment about the x-axis is $\frac{1}{3}y^3\delta x$.

Hence, the second moment of the whole region about the x-axis is given by

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{3}y^3\delta x$$

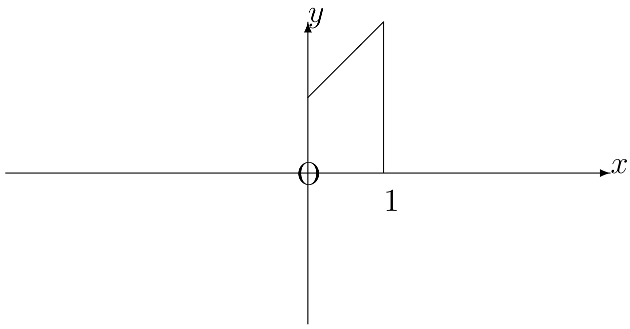
$$= \int_a^b \frac{1}{3}y^3 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the second moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{1}{3}(x+1)^3 \, \mathrm{d}x$$

$$= \frac{1}{3}\int_0^1 (x^3 + 3x^2 + 3x + 1) \, \mathrm{d}x$$

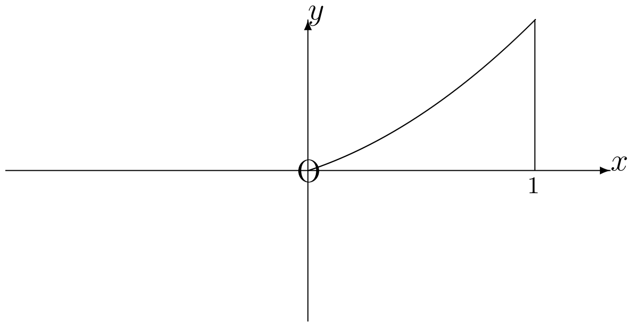
$$= \frac{1}{3}\left[\frac{x^4}{4} + x^3 + + \frac{3x^2}{2} + x\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{4} + 1 + \frac{3}{2} + 1\right) = \frac{5}{4}.$$

2. Determine the second moment about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = xe^x.$$

Solution



$$\text{Second moment } = \int_0^1 \frac{1}{3}x^3e^{3x}\ \mathrm{d}x$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \int_0^1 x^2e^{3x}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \int_0^1 2x\frac{e^{3x}}{3}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \frac{2xe^{3x}}{9} - \frac{2}{3}\int_0^1 \frac{e^{3x}}{3}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{3}\left[x^3\frac{e^{3x}}{3} - x^2\frac{e^{3x}}{3} + \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}\right]_0^1$$

$$= \frac{4e^3+2}{81} \simeq 1.02$$

Note:
The second moment of an area about a certain axis is closely related to its **"moment of inertia"** about that axis.

In fact, for a thin plate with uniform density, ρ, the moment of inertia is ρ times the second moment of area since multiplication by ρ, of elements of area, converts them into elements of mass.