

“JUST THE MATHS”

SLIDES NUMBER

13.10

INTEGRATION APPLICATIONS 10
(Second moments of an arc)

by

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UNIT 13.10 - INTEGRATION APPLICATIONS 10

SECOND MOMENTS OF AN ARC

13.10.1 INTRODUCTION

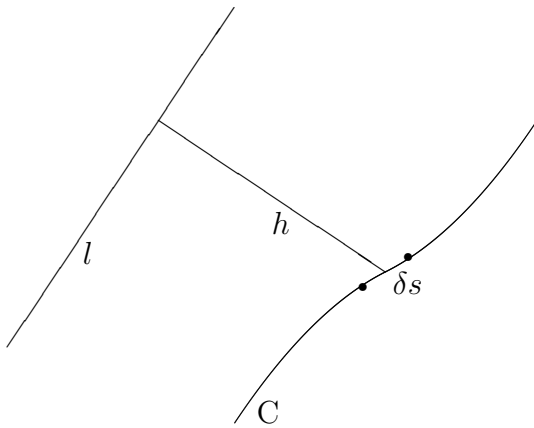
Let C denote an arc (with length s) in the xy -plane of cartesian co-ordinates.

Let δs denote the length of a small element of this arc.

Then the “**second moment**” of C about a fixed line, l , in the plane of C is given by

$$\lim_{\delta s \rightarrow 0} \sum_C h^2 \delta s,$$

where h is the perpendicular distance, from l , of the element with length δs .

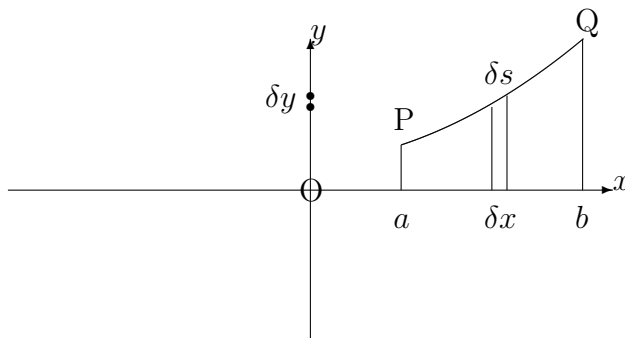


13.10.2 THE SECOND MOMENT OF AN ARC ABOUT THE Y-AXIS

Consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs , by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x -axis) and δy (parallel to the y -axis).

The second moment of each element about the y -axis is $x^2\delta s$.

The total second moment of the arc about the y -axis is given by

$$\lim_{\delta s \rightarrow 0} \sum_C x^2 \delta s.$$

But, by Pythagoras' Theorem

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

Thus, the second moment of the arc becomes

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} x^2 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x \\ = \int_a^b x^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \end{aligned}$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Hence,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\frac{dx}{dt}},$$

provided that $\frac{dx}{dt}$ is positive on the arc being considered.

If $\frac{dx}{dt}$ is negative on the arc, then the above formula needs to be prefixed by a negative sign.

Thus, the second moment of the arc about the y -axis is given by

$$\pm \int_{t_1}^{t_2} x^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as $\frac{dx}{dt}$ is positive or negative.

13.10.3 THE SECOND MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc of the curve whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the second moment about the x -axis will be

$$\int_a^b y^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, the second moment of the arc about the x -axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

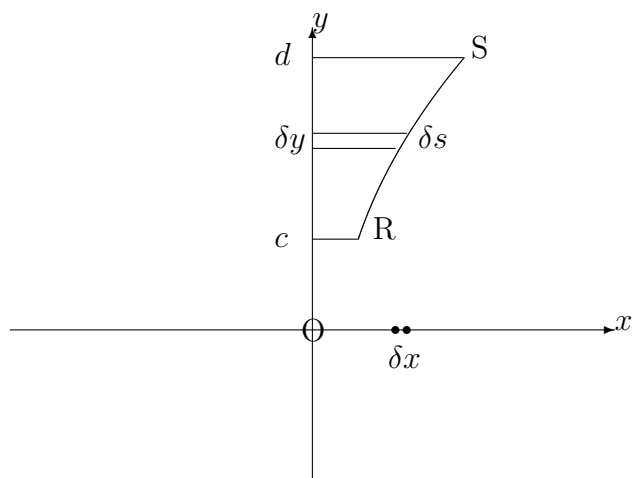
according as $\frac{dx}{dt}$ is positive or negative.

(b) For an arc of the curve whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.10.2 so that the second moment about the x -axis is given by

$$\int_c^d y^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$



Note:

If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, the second moment of the arc about the x -axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

according as $\frac{dy}{dt}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

EXAMPLES

1. Determine the second moments about the x -axis and the y -axis of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

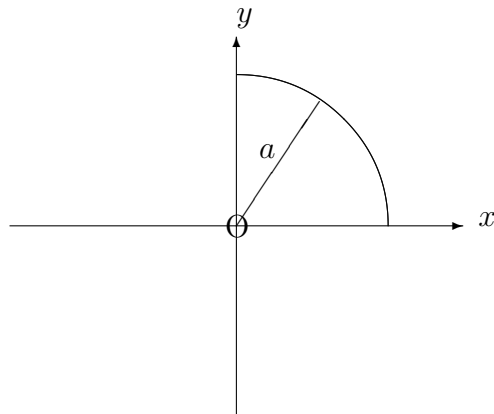
Solution

Using implicit differentiation,

$$2x + 2y \frac{dy}{dx} = 0.$$

Hence,

$$\frac{dy}{dx} = -\frac{x}{y}.$$



The second moment about the y -axis is therefore given by

$$\begin{aligned} & \int_0^a x^2 \sqrt{1 + \frac{x^2}{y^2}} dx \\ &= \int_0^a \frac{x^2}{y} \sqrt{x^2 + y^2} dx. \end{aligned}$$

But

$$x^2 + y^2 = a^2.$$

Therefore,

$$\text{Second moment} = \int_0^a \frac{ax^2}{y} dx.$$

Making the substitution $x = a \sin u$,

$$\begin{aligned}\text{Second moment} &= \int_0^{\frac{\pi}{2}} a^3 \sin^2 u \, du \\ &= a^3 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2u}{2} \, du \\ &= a^3 \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_0^{\frac{\pi}{2}} = \frac{\pi a^3}{4}.\end{aligned}$$

By symmetry, the second moment about the x -axis will also be

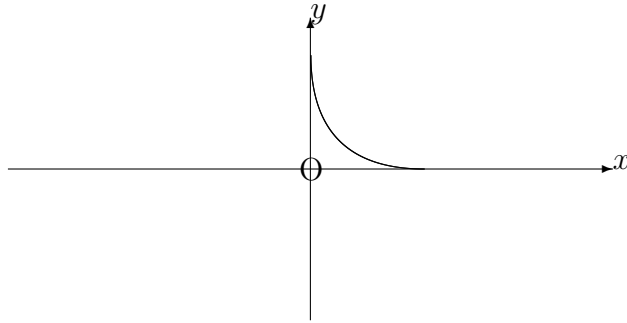
$$\frac{\pi a^3}{4}.$$

2. Determine the second moments about the x -axis and the y -axis of the first quadrant arc of the curve with parametric equations

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta.$$

Solution

$$\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta.$$



The second moment about the y -axis is given by

$$- \int_{\frac{\pi}{2}}^0 x^2 \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} \, d\theta.$$

On using $\cos^2 \theta + \sin^2 \theta \equiv 1$, this becomes

$$\int_0^{\frac{\pi}{2}} a^2 \cos^6 \theta \cdot 3a \cos \theta \sin \theta \, d\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \cos^7 \theta \sin \theta \, d\theta$$

$$= 3a^2 \left[-\frac{\cos^8 \theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8}.$$

Similarly, the second moment about the x -axis is given by

$$\int_0^{\frac{\pi}{2}} y^2 \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

$$\int_0^{\frac{\pi}{2}} a^2 \sin^6 \theta \cdot (3a \cos \theta \sin \theta) \, d\theta$$

$$\begin{aligned}
&= 3a^3 \int_0^{\frac{\pi}{2}} \sin^7 \theta \cos \theta \, d\theta \\
&= 3a^3 \left[\frac{\sin^8 \theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8}.
\end{aligned}$$

This second result could be deduced, by symmetry, from the first.

13.10.4 THE RADIUS OF GYRATION OF AN ARC

Having calculated the second moment of an arc about a certain axis it is possible to determine a positive value, k , with the property that the second moment about the axis is given by sk^2 , where s is the total length of the arc.

We simply divide the value of the second moment by s in order to obtain the value of k^2 and, hence, the value of k .

The value of k is called the “**radius of gyration**” of the given arc about the given axis.

Note:

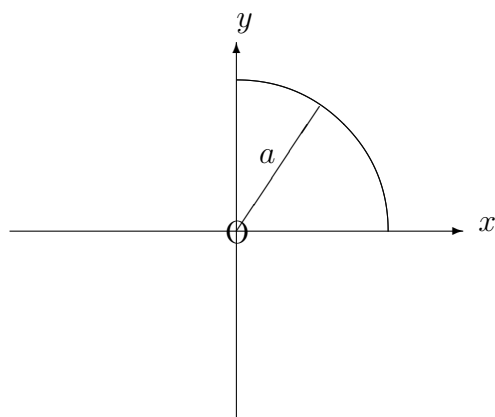
The radius of gyration effectively tries to concentrate the whole arc at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration, about the y -axis, of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution

From Example 1 in Section 13.10.3, the second moment of the arc about the y -axis is equal to

$$\frac{\pi a^3}{4}.$$

Also, the length of the arc is

$$\frac{\pi a}{2}.$$

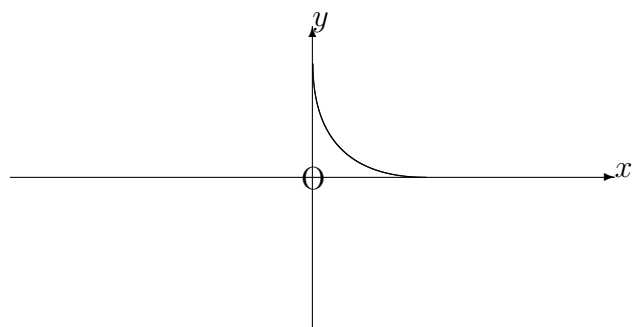
Hence, the radius of gyration is

$$\sqrt{\frac{\pi a^3}{4} \times \frac{2}{\pi a}} = \frac{a}{\sqrt{2}}.$$

2. Determine the radius of gyration, about the y -axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.10.3,

$$\frac{dx}{d\theta} = -3a\cos^2\theta \sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\sin^2\theta \cos\theta.$$

Also, the second moment of the arc about the y -axis is equal to

$$\frac{3a^3}{8}.$$

The length of the arc is given by

$$\begin{aligned} & - \int_{\frac{\pi}{2}}^a \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ & = \int_0^{\frac{\pi}{2}} \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta. \end{aligned}$$

This simplifies to

$$\begin{aligned} & 3a \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta \\ & = 3a \left[\frac{\sin^2 \theta}{2} \right]_0^{\frac{\pi}{2}} = \frac{3a}{2}. \end{aligned}$$

Thus, the radius of gyration is

$$\sqrt{\frac{3a^3}{8} \times \frac{2}{3a}} = \frac{a}{2}.$$