

“JUST THE MATHS”

SLIDES NUMBER

13.1

INTEGRATION APPLICATIONS 1
(The area under a curve)

by

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13.1.1 The elementary formula

13.1.2 Definite integration as a summation

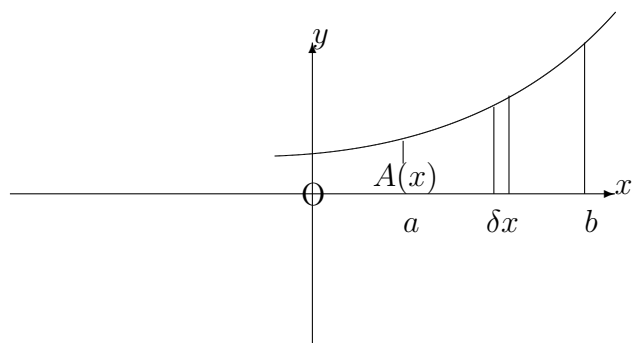
UNIT 13.1 - INTEGRATION APPLICATIONS 1

THE AREA UNDER A CURVE

13.1.1 THE ELEMENTARY FORMULA

We consider, here, the area contained between the x -axis and the arc, from $x = a$ to $x = b$, of the curve $y = f(x)$.

Let $A(x)$ be the area between the curve, the x -axis, the y -axis and the ordinate at some arbitrary value of x .



A small increase of δx in x will lead to a corresponding increase of δA in A .

The increase in A approximates to a narrow rectangle having width δx and height $f(x)$.

Thus,

$$\delta A \simeq f(x)\delta x,$$

which may be written

$$\frac{\delta A}{\delta x} \simeq f(x).$$

Letting δx tend to zero,

$$\frac{dA}{dx} = f(x).$$

Hence, on integrating both sides with respect to x ,

$$A(x) = \int f(x) dx.$$

The constant of integration would need to be such that $A = 0$ when $x = 0$.

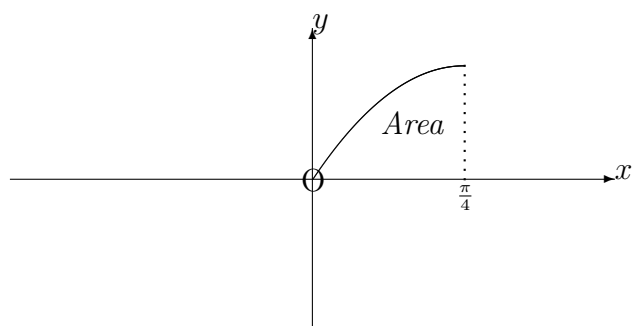
In fact, we do not need to know the constant of integration since

$$A(b) - A(a) = \int_a^b f(x) dx.$$

ILLUSTRATIONS

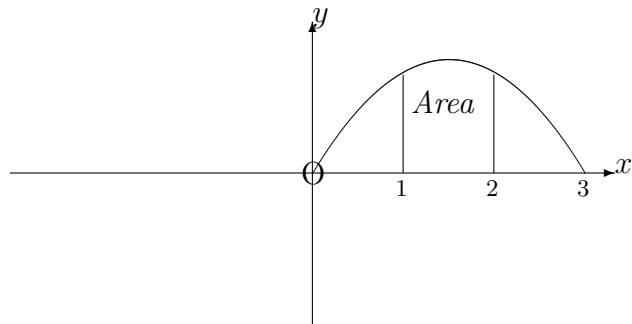
1. The area contained between the x -axis and the curve $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{4}$ is given by

$$\int_0^{\frac{\pi}{4}} \sin 2x \, dx = \left[-\frac{\cos 2x}{2} \right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$



2. The area contained between the x -axis and the curve $y = 3x - x^2$ from $x = 1$ to $x = 2$ is given by

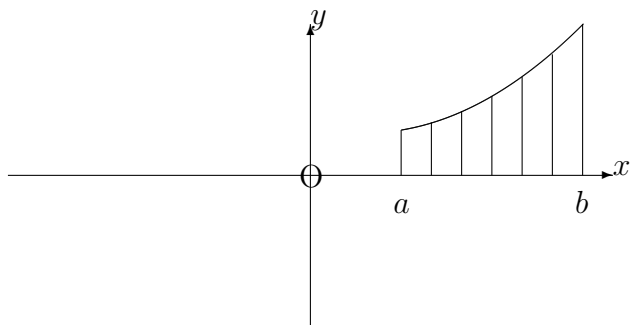
$$\begin{aligned} \int_1^2 (3x - x^2) \, dx &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_1^2 \\ &= \left(6 - \frac{8}{3} \right) - \left(\frac{3}{2} - \frac{1}{3} \right) = \frac{13}{6}. \end{aligned}$$



13.1.2 DEFINITE INTEGRATION AS A SUMMATION

Consider the same area as in the previous section.

We regard the area, now, as the sum (approximately) of a large number of narrow rectangles with typical width δx and typical height $f(x)$.



$$\text{Area} = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} f(x) \delta x.$$

We may conclude that

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)\delta x.$$

Notes:

(i) An area which lies wholly **below** the x -axis will be **negative**.

(ii) If c is any value of x between $x = a$ and $x = b$,

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

(iii) To calculate the TOTAL area between the x -axis and a curve which crosses the x -axis between $x = a$ and $x = b$, account must be taken of any parts of the area which are negative.

(iv) It is a good idea to sketch the area under consideration.

(v) The formula has a wider field of application than the calculation of areas.

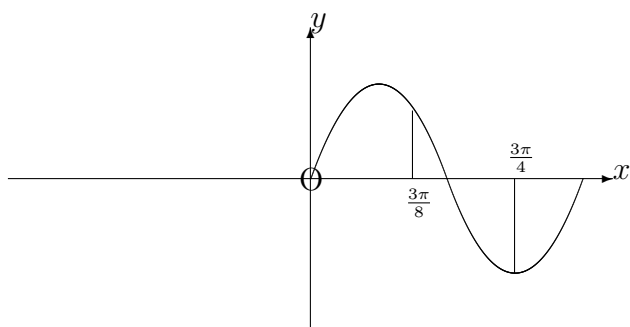
ILLUSTRATIONS

1. The total area between the x -axis and the curve $y = \sin 2x$ from $x = \frac{3\pi}{8}$ and $x = \frac{3\pi}{4}$ is given by

$$\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sin 2x \, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2x \, dx.$$

That is,

$$\left[-\frac{\cos 2x}{2} \right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}} - \left[-\frac{\cos 2x}{2} \right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}$$
$$\left(\frac{1}{2} - \frac{1}{2\sqrt{2}} \right) - \left(0 - \frac{1}{2} \right) = 1 - \frac{1}{2\sqrt{2}}.$$



2. But, the definite integral

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \, dx = \left[-\frac{\cos 2x}{2} \right]_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} = -\frac{1}{2\sqrt{2}}.$$