"JUST THE MATHS"

SLIDES NUMBER

12.9

INTEGRATION 9
(Reduction formulae)

by

A.J.Hobson

UNIT 12.9 - INTEGRATION 9

REDUCTION FORMULAE

INTRODUCTION

For certain integrals, the "integrand" consists of a product involving an unspecified integer, say n;
(for example, $x^n e^x$).

Using integration by parts, it is sometimes possible to express such an integral in terms of a similar integral with n replaced by $(n-1)$ or sometimes $(n-2)$.

The relationship between the two integrals is called a **"reduction formula"**.

By repeated application of a reduction formula, the original integral may be determined in terms of n.

12.9.1 INDEFINITE INTEGRALS

EXAMPLES

1. Obtain a reduction formula for the integral

$$I_n = \int x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$, we obtain

$$I_n = x^n e^x - \int e^x.nx^{n-1}\,\mathrm{d}x.$$

That is,

$$I_n = x^n e^x - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = x^3 e^x - 3I_2,$$

where

$$I_2 = x^2 e^x - 2I_1$$

and

$$I_1 = xe^x - I_0.$$

But

$$I_0 = \int e^x\,\mathrm{d}x = e^x + \text{constant}.$$

Thus,

$$I_3 = x^3 e^x - 3\left[x^2 e^x - 2\left(xe^x - e^x\right)\right] + \text{constant}.$$

That is,

$$I_3 = e^x\left[x^3 - 3x^2 + 6x - 6\right] + C$$

where C is an arbitrary constant.

2. Obtain a reduction formula for the integral

$$I_n = \int x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = x^n \sin x - \int \sin x . n x^{n-1} \, \mathrm{d}x$$

$$= x^n \sin x - n \int x^{n-1} \sin x \, \mathrm{d}x.$$

Using integration by parts again, with $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = x^n \sin x - n \left\{ -x^{n-1} \cos x + \int \cos x . (n-1) x^{n-2} \, \mathrm{d}x \right\}.$$

That is,

$$I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = x^2 \sin x + 2x \cos x - 2I_0,$$

where

$$I_0 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x + C,$$

where C is an arbitrary constant.
Also, substituting $n = 3$,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 3.2.I_1,$$

where

$$I_1 = \int x \cos x \, \mathrm{d}x = x \sin x + \cos x + \text{constant}.$$

Therefore,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 6x \sin x - 6 \cos x + D,$$

where D is an arbitrary constant.

12.9.2 DEFINITE INTEGRALS

EXAMPLES

1. Obtain a reduction formula for the integral

$$I_n = \int_0^1 x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

From the first example of section 12.9.1,

$$I_n = [x^n e^x]_0^1 - nI_{n-1} = e - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = e - 3I_2,$$

where

$$I_2 = e - 2I_1$$

and

$$I_1 = e - I_0.$$

But

$$I_0 = \int_0^1 e^x \, \mathrm{d}x = e - 1.$$

Thus,

$$I_3 = e - 3e + 6e - 6e + 6 = 6 - 2e.$$

2. Obtain a reduction formula for the integral

$$I_n = \int_0^\pi x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

From the second example of section 12.9.1,

$$I_n = \left[x^n \sin x + nx^{n-1} \cos x\right]_0^\pi - n(n-1)I_{n-2}$$

$$= -n\pi^{n-1} - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = -2\pi - 2I_0,$$

where

$$I_0 = \int_0^\pi \cos x \ \mathrm{d}x = [\sin x]_0^\pi = 0.$$

Hence,

$$I_2 = -2\pi.$$

Also, substituting $n = 3$,

$$I_3 = -3\pi^2 - 3.2.I_1,$$

where

$$I_1 = \int_0^\pi x \cos x \ \mathrm{d}x = [x \sin x + \cos x]_0^\pi = -2.$$

Therefore,

$$I_3 = -3\pi^2 + 12.$$