

**“JUST THE MATHS”**

**SLIDES NUMBER**

**12.7**

**INTEGRATION 7**

**(Further trigonometric functions)**

**by**

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**12.7.1 Products of sines and cosines**

**12.7.2 Powers of sines and cosines**

## UNIT 12.7 - INTEGRATION 7

### FURTHER TRIGONOMETRIC FUNCTIONS

#### 12.7.1 PRODUCTS OF SINES AND COSINES

To integrate products of sines cosines, we may use one of the following trigonometric identities:

$$\sin A \cos B \equiv \frac{1}{2} [\sin(A + B) + \sin(A - B)];$$

$$\cos A \sin B \equiv \frac{1}{2} [\sin(A + B) - \sin(A - B)];$$

$$\cos A \cos B \equiv \frac{1}{2} [\cos(A + B) + \cos(A - B)];$$

$$\sin A \sin B \equiv \frac{1}{2} [\cos(A - B) - \cos(A + B)].$$

#### EXAMPLES

1. Determine the indefinite integral

$$\int \sin 2x \cos 5x \, dx.$$

#### Solution

$$\begin{aligned} \int \sin 2x \cos 5x \, dx &= \frac{1}{2} \int [\sin 7x - \sin 3x] \, dx \\ &= -\frac{\cos 7x}{14} + \frac{\cos 3x}{6} + C. \end{aligned}$$

2. Determine the indefinite integral

$$\int \sin 3x \sin x \, dx.$$

**Solution**

$$\begin{aligned} \int \sin 3x \sin x \, dx &= \frac{1}{2} \int [\cos 2x - \cos 4x] \, dx \\ &= \frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C. \end{aligned}$$

## 12.7.2 POWERS OF SINES AND COSINES

We consider the two integrals

$$\int \sin^n x \, dx \quad \text{and} \quad \int \cos^n x \, dx,$$

where  $n$  is a positive integer.

### (a) The Complex Number Method

This method requires us to express  $\cos^n x$  and  $\sin^n x$  as a sum of whole multiples of sines or cosines of whole multiples of  $x$ .

### EXAMPLE

Determine the indefinite integral

$$\int \sin^4 x \, dx.$$

## Solution

By the complex number method,

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4 \cos 2x + 3].$$

## The Working:

$$j^4 2^4 \sin^4 x \equiv \left(z - \frac{1}{z}\right)^4,$$

where  $z \equiv \cos x + j \sin x$ .

That is,

$$16\sin^4 x \equiv z^4 - 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \left(\frac{1}{z}\right)^2 - 4z \cdot \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4;$$

or, after cancelling common factors,

$$16\sin^4 x \equiv z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

which gives

$$16\sin^4 x \equiv 2 \cos 4x - 8 \cos 2x + 6.$$

Hence,

$$\cos^4 x \equiv \frac{1}{8} (\cos 4x - 4 \cos 2x + 3).$$

Therefore,

$$\begin{aligned} \int \sin^4 x \, dx &= \frac{1}{8} \left[ \frac{\sin 4x}{4} - 4 \frac{\sin 2x}{2} + 3x \right] + C \\ &= \frac{1}{32} [\sin 4x - 8 \sin 2x + 12x] + C. \end{aligned}$$

## (b) Odd Powers of Sines and Cosines

The method will be illustrated with examples:

### EXAMPLES

1. Determine the indefinite integral

$$\int \sin^3 x \, dx.$$

#### Solution

$$\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx.$$

That is,

$$\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

$$\begin{aligned}
&= \int (\sin x - \cos^2 x \cdot \sin x) \, dx \\
&= -\cos x + \frac{\cos^3 x}{3} + C.
\end{aligned}$$

2. Determine the indefinite integral

$$\int \cos^7 x \, dx.$$

### **Solution**

$$\int \cos^7 x \, dx = \int \cos^6 x \cdot \cos x \, dx.$$

That is,

$$\begin{aligned}
\int \cos^7 x \, dx &= \int (1 - \sin^2 x)^3 \cdot \cos x \, dx \\
&= \int (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \cdot \cos x \, dx \\
&= \sin x - \sin^3 x + 3 \cdot \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.
\end{aligned}$$

### **(c) Even Powers of Sines and Cosines**

This method is tedious if the even power is higher than 4.

In such cases, it is best to use the complex number method.

In the examples which follow, we shall need the trigonometric identity

$$\cos 2A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1.$$

## EXAMPLES

1. Determine the indefinite integral

$$\int \sin^2 x \, dx.$$

### Solution

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx.$$

### Solution

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos^2 x \, dx &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \, dx \\ &= \left[ \frac{x}{2} + \frac{\sin 2x}{4} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \end{aligned}$$

3. Determine the indefinite integral

$$\int \cos^4 x \, dx.$$

**Solution**

$$\int \cos^4 x \, dx = \int [\cos^2 x]^2 \, dx = \int \left[ \frac{1}{2}(1 + \cos 2x) \right]^2 \, dx.$$

That is,

$$\int \cos^4 x \, dx = \int \frac{1}{4} (1 + 2 \cos 2x + \cos^2 2x) \, dx$$

$$= \int \frac{1}{4} \left( 1 + 2 \cos 2x + \frac{1}{2}[1 + \cos 4x] \right) \, dx$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$