"JUST THE MATHS"

SLIDES NUMBER

12.6

INTEGRATION 6
(Integration by partial fractions)

by

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12.6.1 Introduction and illustrations

UNIT 12.6 - INTEGRATION 6
INTEGRATION BY PARTIAL FRACTIONS

12.6.1 INTRODUCTION AND ILLUSTRATIONS

The following results will cover most elementary problems involving partial fractions:

RESULTS

1.

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln(ax+b) + C.$$

2.

$$\int \frac{1}{(ax+b)^n}\,\mathrm{d}x = \frac{1}{a}.\frac{(ax+b)^{-n+1}}{-n+1} + C \text{ provided } n \neq 1.$$

3.

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

4.

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

Alternatively, if hyperbolic functions have been studied,

$$\int \frac{1}{a^2 - x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C.$$

5.

$$\int \frac{2ax + b}{ax^2 + bx + c}\,\mathrm{d}x = \ln(ax^2 + bx + c) + C.$$

ILLUSTRATIONS

We use some of the results of examples on partial fractions in Unit 1.8

1.

$$\int \frac{7x + 8}{(2x + 3)(x - 1)}\,\mathrm{d}x = \int \left[\frac{1}{2x + 3} + \frac{3}{x - 1}\right]\,\mathrm{d}x$$

$$= \frac{1}{2}\ln(2x + 3) + 3\ln(x - 1) + C.$$

2.

$$\int_6^8 \frac{3x^2 + 9}{(x - 5)(x^2 + 2x + 7)}\,\mathrm{d}x = \int_6^8 \left[\frac{2}{x - 5} + \frac{x + 1}{x^2 + 2x + 7}\right]\,\mathrm{d}x$$

$$= \left[2\ln(x - 5) + \frac{1}{2}\ln(x^2 + 2x + 7)\right]_6^8 \simeq 2.427$$

3.

$$\int \frac{9}{(x+1)^2(x-2)} = \int \left[\frac{-1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}\right] \mathrm{d}x$$

$$= -\ln(x+1) + \frac{3}{x+1} + \ln(x-2) + C.$$

4.

$$\int \frac{4x^2+x+6}{(x-4)(x^2+4x+5)} \,\mathrm{d}x = \int \left[\frac{2}{x-4} + \frac{2x+1}{x^2+4x+5}\right] \mathrm{d}x.$$

The second partial fraction has a numerator of $2x+1$ which is not the derivative of x^2+4x+5;

but we simply rearrange as

$$\frac{(2x+4)-3}{x^2+4x+5} \equiv \frac{2x+4}{x^2+4x+5} - \frac{3}{(x+2)^2+1}.$$

By Unit 12.3,

Answer $= 2\ln(x-4)+\ln(x^2+4x+5)-3\tan^{-1}(x+2)+C.$