“JUST THE MATHS”

SLIDES NUMBER

12.5

INTEGRATION 5
(Integration by parts)

by

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12.5.1 The standard formula

UNIT 12.5 - INTEGRATION 5

INTEGRATION BY PARTS

12.5.1 THE STANDARD FORMULA

The method described here is for integrating the product of two functions.

It is possible to develop a suitable formula by considering, instead, the **derivative** of the product of two functions.

We consider, first, the following comparison:

$\frac{d}{dx}[x\sin x] = x\cos x + \sin x$	$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$
$x\cos x = \frac{d}{dx}[x\sin x] - \sin x$	$u\frac{dv}{dx} = \frac{d}{dx}[uv] - v\frac{du}{dx}$
$\int x\cos x \ dx = x\sin x - \int \sin x \ dx$	$\int u\frac{dv}{dx} \ dx = uv - \int v\frac{du}{dx} \ dx$
$= x\sin x + \cos x + C$	

By labelling the product of two given functions as

$$u\frac{dv}{dx},$$

we may express the given integral in terms of another integral (hopefully simpler than the original).

The formula for **“integration by parts”** is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

EXAMPLES

1. Determine the indefinite integral

$$I = \int x^2 e^{3x}\,\mathrm{d}x.$$

Solution

In theory, it does not matter which element of the product $x^2 e^{3x}$ is labelled as u and which is labelled as $\frac{\mathrm{d}v}{\mathrm{d}x}$.

In this case, we shall take

$$u = x^2 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Hence,

$$I = x^2\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.2x\,\mathrm{d}x.$$

That is,

$$I = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int xe^{3x}\,\mathrm{d}x.$$

Using integration by parts a second time, we shall set

$$u = x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Thus,

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.1 \, \mathrm{d}x\right].$$

The integration may now be completed to obtain

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C,$$

or

$$I = \frac{e^{3x}}{27}\left[9x^2 - 6x + 2\right] + C.$$

2. Determine the indefinite integral

$$I = \int x \ln x \, \mathrm{d}x.$$

Solution

In this case, we choose

$$u = \ln x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = x,$$

obtaining

$$I = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x} \, \mathrm{d}x.$$

That is,

$$I = \frac{1}{2}x^2 \ln x - \int \frac{x}{2} \, \mathrm{d}x.$$

Hence,

$$I = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C.$$

3. Determine the indefinite integral

$$I = \int \ln x \, \mathrm{d}x.$$

Solution

Let

$$u = \ln x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = x \ln x - \int x . \frac{1}{x} \, \mathrm{d}x,$$

giving

$$I = x \ln x - x + C.$$

4. Evaluate the definite integral

$$I = \int_0^1 \sin^{-1} x \, \mathrm{d}x.$$

Solution

Let

$$u = \sin^{-1} x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = [x\sin^{-1}x]_0^1 - \int_0^1 x.\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

That is,

$$I = [x\sin^{-1}x + \sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1.$$

5. Determine the indefinite integral

$$I = \int e^{2x}\cos x\,\mathrm{d}x.$$

Solution

We shall set

$$u = e^{2x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x.$$

Hence,

$$I = e^{2x}\sin x - \int (\sin x).2e^{2x}\,\mathrm{d}x.$$

That is,

$$I = e^{2x}\sin x - 2\int e^{2x}\sin x\,\mathrm{d}x.$$

Now we integrate by parts again, setting

$$u = e^{2x} \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x.$$

Therefore,

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x - \int (-\cos x).2e^{2x}\, \mathrm{d}x\right].$$

The original integral has appeared again on the right hand side to give

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2I\right].$$

On simplification,

$$5I = e^{2x}\sin x + 2e^{2x}\cos x,$$

so that

$$I = \frac{1}{5}e^{2x}[\sin x + 2\cos x] + C.$$

Priority Order for choosing u

1. LOGARITHMS or INVERSE FUNCTIONS;

2. POWERS OF x;

3. POWERS OF e.