“JUST THE MATHS”

SLIDES NUMBER

12.4

INTEGRATION 4
(Integration by substitution in general)

by

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UNIT 12.4 - INTEGRATION 4 INTEGRATION BY SUBSTITUTION IN GENERAL

12.4.1 EXAMPLES USING THE STANDARD FORMULA

With any integral

$$\int f(x)\mathrm{d}x$$

we may wish to substitute for x in terms of a new variable, u.

From Unit 12.1,

$$\int f(x)\mathrm{d}x = \int f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u.$$

This result was originally used for Functions of a Linear Function.

For this Unit, substitutions other than linear ones will be illustrated.

EXAMPLES

1. Use the substitution $x = a \sin u$ to show that

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$$

Solution

We shall assume that u is the **acute** angle for which $x = a \sin u$.

In effect, we substitute $u = \sin^{-1}\frac{x}{a}$ using the **principal** value of the inverse function.

If $x = a \sin u$, then $\frac{\mathrm{d}x}{\mathrm{d}u} = a \cos u$ so that the integral becomes

$$\int \frac{a \cos u}{\sqrt{a^2 - a^2 \sin^2 u}} \mathrm{d}u.$$

But, from trigonometric identities,

$$\sqrt{a^2 - a^2 \sin^2 u} \equiv a \cos u,$$

both sides being positive when u is an acute angle.

Thus, we have

$$\int 1 \mathrm{d}u = u + C = \sin^{-1}\frac{x}{a} + C.$$

2. Use the substitution $u = \frac{1}{x}$ to determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{x\sqrt{1+x^2}}.$$

Solution

Writing

$$x = \frac{1}{u},$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}.$$

Hence,

$$z = \int \frac{1}{\frac{1}{u}\sqrt{1+\frac{1}{u^2}}}. - \frac{1}{u^2}\mathrm{d}u.$$

That is,

$$z = \int -\frac{1}{\sqrt{u^2+1}} = -\ln(u + \sqrt{u^2+1}) + C.$$

Thus,

$$z = -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) + C.$$

12.4.2 INTEGRALS INVOLVING A FUNCTION AND ITS DERIVATIVE

Two useful results:

(a)

$$\int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

provided $n \neq -1$.

(b)

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x) + C.$$

These two results are obtained from the substitution

$$u = f(x).$$

In both cases,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = f'(x).$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{f'(x)}.$$

This converts the integrals, respectively, into

(a)

$$\int u^n \mathrm{d}u = \frac{u^{n+1}}{n+1} + C,$$

and

(b)

$$\int \frac{1}{u} \mathrm{d}u = \ln u + C.$$

EXAMPLES

1. Evaluate the definite integral
$$\int_0^{\frac{\pi}{3}} \sin^3 x . \cos x \ \mathrm{d}x.$$
Solution
In this example, we can consider $\sin x$ to be $f(x)$ and $\cos x$ to be $f'(x)$.
Thus, by result (a),
$$\int_0^{\frac{\pi}{3}} \sin^3 x . \cos x \mathrm{d}x = \left[\frac{\sin^4 x}{4} \right]_0^{\frac{\pi}{3}} = \frac{9}{64},$$
using $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.
2. Integrate the function
$$\frac{2x+1}{x^2 + x - 11}$$

with respect to x.

Solution

Here, we can identify x^2+x-11 with $f(x)$ and $2x+1$ with $f'(x)$.

Thus, by result (b),

$$\int \frac{2x+1}{x^2+x-11}\mathrm{d}x = \ln(x^2+x-11)+C.$$