“JUST THE MATHS”

SLIDES NUMBER

12.3

INTEGRATION 3
(The method of completing the square)

by

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UNIT 12.3 - INTEGRATION 3 THE METHOD OF COMPLETING THE SQUARE

12.3.1 INTRODUCTION AND EXAMPLES

A substitution such as $u = \alpha x + \beta$ may also be used with integrals of the form

$$\int \frac{1}{px^2 + qx + r} \mathrm{d}x$$

and

$$\int \frac{1}{\sqrt{px^2 + qx + r}} \mathrm{d}x.$$

Note:

These may also be written

$$\int \frac{\mathrm{d}x}{px^2 + qx + r}$$

and

$$\int \frac{\mathrm{d}x}{\sqrt{px^2 + qx + r}}.$$

Standard Results To Use

1.

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

2.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} + C.$$

3.

$$\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \sinh^{-1}\frac{x}{a}+C \text{ or } \ln(x+\sqrt{x^2+a^2})+C.$$

4.

$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \cosh^{-1}\frac{x}{a}+C \text{ or } \ln(x+\sqrt{x^2-a^2})+C.$$

5.

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C;$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

EXAMPLES

1. Determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{\sqrt{x^2 + 2x - 3}}.$$

Solution

Completing the square,

$$x^2 + 2x - 3 \equiv (x+1)^2 - 4 \equiv (x+1)^2 - 2^2.$$

Hence,

$$z = \int \frac{\mathrm{d}x}{\sqrt{(x+1)^2 - 2^2}}.$$

Putting $u = x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.
Thus,

$$z = \int \frac{\mathrm{d}u}{\sqrt{u^2 - 2^2}},$$

giving

$$z = \ln\left[u + \sqrt{u^2 - 2^2}\right] + C.$$

Returning to the variable x,

$$z = \ln\left[x + 1 + \sqrt{x^2 + 2x - 3}\right] + C.$$

2. Evaluate the definite integral

$$z = \int_3^7 \frac{\mathrm{d}x}{x^2 - 6x + 25}.$$

Solution

Completing the square,

$$x^2 - 6x + 25 \equiv (x-3)^2 + 16.$$

Hence,

$$z = \int_3^7 \frac{\mathrm{d}x}{(x-3)^2 + 16}.$$

Putting $u = x - 3$, we obtain $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.
Thus,

$$z = \int_0^4 \frac{\mathrm{d}u}{u^2 + 16},$$

giving

$$z = \left[\frac{1}{4}\tan^{-1}\frac{u}{4}\right]_0^4 = \frac{\pi}{16}.$$

Alternatively,

$$z = \left[\frac{1}{4}\tan^{-1}\frac{x-3}{4}\right]_3^7 = \frac{\pi}{16}.$$

Note:

In cases where $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, we may treat the linear expression within the completed square as if it were a single x, then write the result straight down !