

“JUST THE MATHS”

SLIDES NUMBER

12.10

INTEGRATION 10
(Further reduction formulae)

by

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UNIT 12.10 - INTEGRATION 10

FURTHER REDUCTION FORMULAE

INTRODUCTION

There are two definite integrals which are worthy of special consideration.

They are

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, dx.$$

First, we shall establish the reduction formulae for the equivalent indefinite integrals.

12.10.1 INTEGER POWERS OF A SINE

Suppose that

$$I_n = \int \sin^n x \, dx.$$

Writing the integrand as the product of two functions,

$$I_n = \int \sin^{n-1} x \sin x \, dx.$$

Using integration by parts, with

$$u = \sin^{n-1}x \quad \text{and} \quad \frac{dv}{dx} = \sin x,$$

$$I_n = \sin^{n-1}x(-\cos x) + \int (n-1)\sin^{n-2}x\cos^2x \, dx.$$

But, $\cos^2x \equiv 1 - \sin^2x$.

Hence,

$$I_n = -\sin^{n-1}x \cos x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n} [-\sin^{n-1}x \cos x + (n-1)I_{n-2}].$$

EXAMPLE

Determine the indefinite integral

$$\int \sin^6x \, dx.$$

Solution

$$I_6 = \frac{1}{6} [-\sin^5x \cos x + 5I_4].$$

But,

$$I_4 = \frac{1}{4} [-\sin^3 x \cos x + 3I_2],$$

$$I_2 = \frac{1}{2} [-\sin x \cos x + I_0]$$

and

$$I_0 = \int dx = x + A.$$

Hence,

$$I_2 = \frac{1}{2} [-\sin x \cos x + x + A];$$

$$I_4 = \frac{1}{4} \left[-\sin^3 x \cos x - \frac{3}{2} \sin x \cos x + \frac{3}{2}x + B \right];$$

$I_6 =$

$$\frac{1}{6} \left[-\sin^5 x \cos x - \frac{5}{4} \sin^3 x \cos x - \frac{15}{8} \sin x \cos x + \frac{15}{8}x + C \right]$$

$$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5x}{16} + D.$$

12.10.2 INTEGER POWERS OF A COSINE

Suppose that

$$I_n = \int \cos^n x \, dx.$$

Writing the integrand as the product of two functions,

$$I_n = \int \cos^{n-1} x \cos x \, dx.$$

Using integration by parts, with

$$u = \cos^{n-1} x \quad \text{and} \quad \frac{dv}{dx} = \cos x,$$

$$I_n = \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \, dx.$$

But, $\sin^2 x \equiv 1 - \cos^2 x$.

Hence,

$$I_n = \cos^{n-1} x \sin x + (n-1)[I_{n-2} - I_n],$$

$$\text{or } I_n = \frac{1}{n} [\cos^{n-1} x \sin x + (n-1)I_{n-2}].$$

EXAMPLE

Determine the indefinite integral

$$\int \cos^5 x \, dx.$$

Solution

$$I_5 = \frac{1}{5} [\cos^4 x \sin x + 4I_3].$$

But,

$$I_3 = \frac{1}{3} [\cos^2 x \sin x + 2I_1]$$

and

$$I_1 = \int \cos x \, dx = \sin x + A.$$

Hence,

$$I_3 = \frac{1}{3} [\cos^2 x \sin x + 2 \sin x + B];$$

$$\begin{aligned} I_5 &= \frac{1}{5} \left[\cos^4 x \sin x + \frac{4}{3} \cos^2 x \sin x + \frac{8}{3} \sin x + C \right] \\ &= \frac{1}{5} \cos^4 x \sin x + \frac{4}{15} \cos^2 x \sin x + \frac{8}{15} \sin x + D. \end{aligned}$$

12.10.3 WALLIS'S FORMULAE

Here, we consider the definite integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, dx.$$

Denoting either of these integrals by I_n , the reduction formula (in both cases) reduces to

$$I_n = \frac{n-1}{n} I_{n-2}.$$

There are two versions of “**Wallis’s formulae**”

(a) n is an odd number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \cdots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1.$$

But,

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx \quad \text{or} \quad I_1 = \int_0^{\frac{\pi}{2}} \cos x \, dx,$$

both of which have a value of 1.

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5)\dots 6.4.2}{n(n-2)(n-4)\dots 7.5.3}.$$

(b) n is an even number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0.$$

But

$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5)\dots 5.3.1}{n(n-2)(n-4)\dots 6.4.2} \frac{\pi}{2}.$$

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, dx.$$

Solution

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, dx = \frac{4.2}{5.3} = \frac{8}{15}.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^4 x \, dx = \frac{3.1 \pi}{4.2 \cdot 2} = \frac{3\pi}{16}.$$

12.10.4 COMBINATIONS OF SINES AND COSINES

Wallis's formulae may be applied to integrals of the form

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx,$$

where either m or n (or both) is an even number.

We use $\sin^2 x \equiv 1 - \cos^2 x$ or $\cos^2 x \equiv 1 - \sin^2 x$.

EXAMPLE

Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, dx.$$

Solution

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, dx \\ &= \int_0^{\frac{\pi}{2}} \cos^5 x (1 - \cos^2 x) \, dx \\ &= \int_0^{\frac{\pi}{2}} (\cos^5 x - \cos^7 x) \, dx. \end{aligned}$$

This may be interpreted as

$$I_5 - I_7 = \frac{4.2}{5.3} - \frac{5.4.3}{6.4.2} = \frac{8}{15} - \frac{16}{35} = \frac{8}{105}.$$