

“JUST THE MATHS”

SLIDES NUMBER

11.6

DIFFERENTIATION APPLICATIONS 6
(Small increments and small errors)

by

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11.6.1 Small increments

11.6.2 Small errors

UNIT 11.6

DIFFERENTIATION APPLICATIONS 6

SMALL INCREMENTS AND SMALL ERRORS

11.6.1 SMALL INCREMENTS

If

$$y = f(x),$$

suppose that x is subject to a small “**increment**”, δx .

“Increment” means that δx is positive when x is **increased**, but negative when x is **decreased**.

The exact value of the corresponding increment, δy , in y is given by

$$\delta y = f(x + \delta x) - f(x).$$

This can often be difficult to evaluate.

However, since δx is small,

$$\frac{f(x + \delta x) - f(x)}{\delta x} \simeq \frac{dy}{dx}.$$

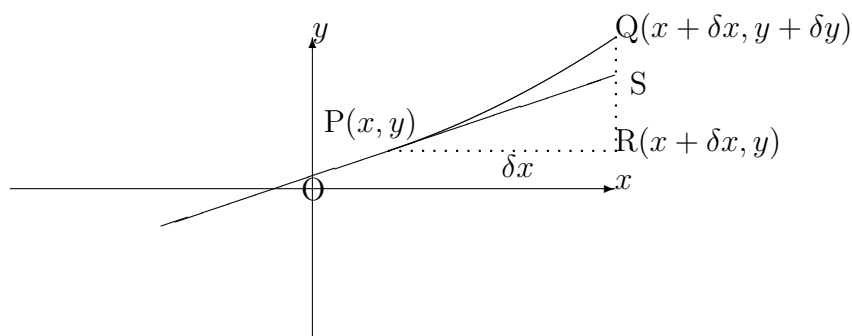
That is,

$$\frac{\delta y}{\delta x} \simeq \frac{dy}{dx}.$$

Thus,

$$\delta y \simeq \frac{dy}{dx} \delta x.$$

Diagrammatic approach



$PR = \delta x$, $QR = \delta y$ and the gradient of PS is the value of $\frac{dy}{dx}$ at P .

Taking SR as an approximation to QR ,

$$\frac{SR}{PR} = \left[\frac{dy}{dx} \right]_P.$$

$$\frac{\text{SR}}{\delta x} = \left[\frac{dy}{dx} \right]_P.$$

Hence,

$$\delta y \simeq \left[\frac{dy}{dx} \right]_P \delta x.$$

Notes:

(i) $\frac{dy}{dx} \delta x$ is the “**total differential of y** ” (or simply the “differential of y ”).

(ii) It is important **not** to use “differential” when referring to a “derivative”.

The correct alternative to “derivative” is “differential coefficient”.

(iii) A more rigorous calculation of δy comes from the result known as “Taylor’s Theorem”:

$$f(x+\delta x) = f(x) + f'(x)\delta x + \frac{f''(x)}{2!}(\delta x)^2 + \frac{f'''(x)}{3!}(\delta x)^3 + \dots$$

Hence, if δx is small enough for powers of two and above to be neglected, then

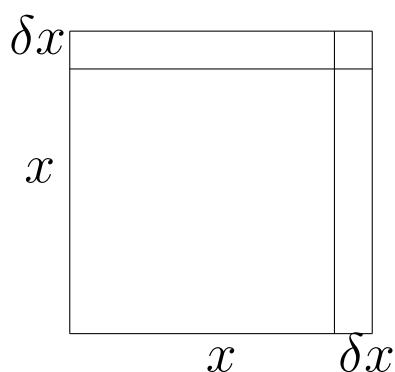
$$f(x + \delta x) - f(x) \simeq f'(x)\delta x.$$

EXAMPLES

1. If a square has side x cms., obtain both the exact and the approximate values of the increment in the area A cms². when x is increased by δx .

Solution

(a) Exact Method



The area is given by the formula

$$A = x^2.$$

$$A + \delta A = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2.$$

That is,

$$\delta A = 2x\delta x + (\delta x)^2.$$

(b) Approximate Method

Here, we use

$$\frac{dA}{dx} = 2x$$

to give

$$\delta A \simeq 2x\delta x.$$

From the diagram, that the two results differ only by the area of the small square, with side δx .

2. If

$$y = xe^{-x},$$

calculate approximately the change in y when x increases from 5 to 5.03.

Solution

We have

$$\frac{dy}{dx} = e^{-x}(1 - x),$$

so that

$$\delta y \simeq e^{-x}(1 - x)\delta x,$$

where $x = 5$ and $\delta x = 0.03$.

Hence,

$$\delta y \simeq e^{-5} \cdot (1 - 5) \cdot (0.03) \simeq -0.00809$$

Thus, y **decreases** by approximately 0.00809

Note:

The exact value is given by

$$\delta y = 5.3e^{-5.3} - 5e^{-5} \simeq -0.00723$$

3. If

$$y = xe^{-x},$$

determine, in terms of x , the percentage change in y when x is increased by 2%.

Solution

Once again, we have

$$\delta y = e^{-x}(1 - x)\delta x;$$

but, this time, $\delta x = 0.02x$, so that

$$\delta y = e^{-x}(1 - x) \times 0.02x.$$

The **percentage** change in y is given by

$$\frac{\delta y}{y} \times 100 = \frac{e^{-x}(1 - x) \times 0.02x}{xe^{-x}} \times 100 = 2(1 - x).$$

That is, y increases by $2(1-x)\%$, which will be positive when $x < 1$ and negative when $x > 1$.

11.6.2 SMALL ERRORS

If

$$y = f(x),$$

suppose that x is known to be subject to an error in measurement.

In particular, suppose x is known to be **too large** by a small amount δx .

The correct value of x could be obtained if we **decreased** it by δx .

That is, if we **increased** it by $-\delta x$.

Correspondingly, the value of y will **increase** by approximately $-\frac{dy}{dx}\delta x$.

That is, y will **decrease** by approximately $\frac{dy}{dx}\delta x$.

Summary

If x is too large by an amount δx , then y is too large by approximately $\frac{dy}{dx}\delta x$.

Note:

If $\frac{dy}{dx}$ itself is negative, y will be too small when x is too large and vice versa.

EXAMPLES

1. If

$$y = x^2 \sin x,$$

calculate, approximately, the error in y when x is measured as 3, but this measurement is subsequently discovered to be too large by 0.06.

Solution

We have

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$$

and, hence,

$$\delta y \simeq (x^2 \cos x + 2x \sin x) \delta x,$$

where $x = 3$ and $\delta x = 0.06$.

Thus,

$$\delta y \simeq (3^2 \cos 3 + 6 \sin 3) \times 0.06 \simeq -0.4838$$

That is, y is too small by approximately 0.4838.

2. If

$$y = \frac{x}{1+x},$$

determine approximately, in terms of x , the percentage error in y when x is subject to an error of 5%.

Solution

We have

$$\frac{dy}{dx} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2},$$

so that

$$\delta y \simeq \frac{1}{(1+x)^2} \delta x,$$

where $\delta x = 0.05x$.

The **percentage** error in y is thus given by

$$\frac{\delta y}{y} \times 100 \simeq \frac{1}{(1+x)^2} \times 0.05x \times \frac{x+1}{x} \times 100 = \frac{5}{1+x}.$$

Hence, y is too large by approximately $\frac{5}{1+x}\%$ which will be positive when $x > -1$ and negative when $x < -1$.