"JUST THE MATHS"

SLIDES NUMBER

11.5

DIFFERENTIATION APPLICATIONS 5
(Maclaurin's and Taylor's series)

by

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UNIT 11.5 - DIFFERENTIATION APPLICATIONS 5

MACLAURIN'S AND TAYLOR'S SERIES

11.5.1 MACLAURIN'S SERIES

The problem here is to approximate, to a polynomial, functions which are not already in polynomial form.

THE GENERAL THEORY

Let $f(x)$ be a given function of x which is not a polynomial.

Assume that $f(x)$ may be expressed as an infinite "power series".

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ \ldots$$

To justify this assumption, we must determine the **"coefficients"**, a_0, a_1, a_2, a_3, a_4,.........

This is possible as an application of differentiation.

(a) Firstly, if we substitute $x = 0$ into the assumed formula for $f(x)$, we obtain $f(0) = a_0$ so that

$$a_0 = f(0).$$

(b) Secondly, if we differentiate the assumed formula for $f(x)$ once with respect to x,

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

On substituting $x = 0$, $f'(0) = a_1$ so that

$$a_1 = f'(0).$$

(c) Differentiating a second time,

$$f''(x) = 2a_2 + (3 \times 2)a_3x + (4 \times 3)a_4x^2 + \ldots$$

On substituting $x = 0$, $f''(0) = 2a_2$ so that

$$a_2 = \frac{1}{2}f''(0).$$

(d) Differentiating a third time,

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4x + \ldots$$

On substituting $x = 0$, $f'''(0) = (3 \times 2)a_3$ so that

$$a_3 = \frac{1}{3!}f'''(0).$$

(e) Continuing this process leads to the general formula

$$a_n = \frac{1}{n!} f^{(n)}(0),$$

where $f^{(n)}(0)$ means the value, at $x = 0$, of the n-th derivative of $f(x)$.

Summary

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

This is called the **"Maclaurin's series for** $f(x)$".

Notes:

(i) We assume that all of the derivatives of $f(x)$ exist at $x = 0$; otherwise the above result is invalid.

(ii) The Maclaurin's series for a particular function may not be used when the series diverges.

(iii) If x is small enough to neglect powers of x after the n-th power, then Maclaurin's series approximates $f(x)$ to a polynomial of degree n.

11.5.2 STANDARD SERIES

The ranges of values of x for which the results are valid will be stated without proof.

1. The Exponential Series

(i) $f(x) \equiv e^x$; hence, $f(0) = e^0 = 1$.

(ii) $f'(x) = e^x$; hence, $f'(0) = e^0 = 1$.

(iii) $f''(x) = e^x$; hence, $f''(0) = e^0 = 1$.

(iv) $f'''(x) = e^x$; hence, $f'''(0) = e^0 = 1$.

(v) $f^{(iv)}(x) = e^x$; hence, $f^{(iv)}(0) = e^0 = 1$.

Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots$$

It may be shown that this series is valid for all values of x.

2. The Sine Series

(i) $f(x) \equiv \sin x$; hence, $f(0) = \sin 0 = 0$.

(ii) $f'(x) = \cos x$; hence, $f'(0) = \cos 0 = 1$.

(iii) $f''(x) = -\sin x$; hence, $f''(0) = -\sin 0 = 0$.

(iv) $f'''(x) = -\cos x$; hence, $f'''(0) = -\cos 0 = -1$.

(v) $f^{(iv)}(x) = \sin x$; hence, $f^{(iv)}(0) = \sin 0 = 0$.

(vi) $f^{(v)}(x) = \cos x$; hence, $f^{(v)}(0) = \cos 0 = 1$.

Thus,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$

It may be shown that this series is valid for all values of x.

3. The Cosine Series

(i) $f(x) \equiv \cos x$; hence, $f(0) = \cos 0 = 1$.

(ii) $f'(x) = -\sin x$; hence, $f'(0) = -\sin 0 = 0$.

(iii) $f''(x) = -\cos x$; hence, $f''(0) = -\cos 0 = -1$.

(iv) $f'''(x) = \sin x$; hence, $f'''(0) = \sin 0 = 0$.

(v) $f^{(iv)}(x) = \cos x$; hence, $f^{(iv)}(0) = \cos 0 = 1$.

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots$$

It may be shown that this series is valid for all values of x.

4. The Logarithmic Series

It is not possible to find a Maclaurin's series for the function $\ln x$ since neither the function nor its derivatives exist at $x = 0$.

As an alternative, we may consider the function $\ln(1+x)$

(i) $f(x) \equiv \ln(1+x)$; hence, $f(0) = \ln 1 = 0$.

(ii) $f'(x) = \frac{1}{1+x}$; hence, $f'(0) = 1$.

(iii) $f''(x) = -\frac{1}{(1+x)^2}$; hence, $f''(0) = 1$.

(iv) $f'''(x) = \frac{2}{(1+x)^3}$; hence, $f'''(0) = 2$.

(v) $f^{(iv)}(x) = -\frac{2\times 3}{(1+x)^4}$; hence, $f^{(iv)}(0) = -(2\times 3)$.

Thus,

$$\ln(1+x) = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - (2\times 3)\frac{x^4}{4!} + \ \ldots$$

which simplifies to

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots$$

It may be shown that this series is valid for $-1 < x \leq 1$.

5. The Binomial Series

When n is a positive integer, the expansion of $(1+x)^n$ in ascending powers of x is a **finite** series obtainable, for example, by Pascal's Triangle.

In all other cases, the series is **infinite** as follows:

(i) $f(x) \equiv (1+x)^n$; hence, $f(0) = 1$

(ii) $f'(x) = n(1+x)^{n-1}$; hence, $f'(0) = n$.

(iii) $f''(x) = n(n-1)(1+x)^{n-2}$;
hence, $f''(0) = n(n-1)$.

(iv) $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$;
hence, $f'''(0) = n(n-1)(n-2)$.

(v) $f^{(iv)}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$;
hence, $f^{(iv)}(0) = n(n-1)(n-2)(n-3)$.

Thus,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 +$$

$$\frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

If n is a positive integer, all of the derivatives of $(1+x)^n$ after the n-th derivative are identically equal to zero; so the series is a finite series ending with the term in x^n.

In all other cases, the series is an infinite series and it may be shown that it is valid whenever $-1 < x \leq 1$.

EXAMPLES

1. Use the Maclaurin's series for $\sin x$ to evaluate
$$\lim_{x\to 0} \frac{x+\sin x}{x(x+1)}.$$

   Solution

   Substituting the series for $\sin x$ gives
$$\lim_{x\to 0} \frac{x+x-\frac{x^3}{3!}+\frac{x^5}{5!}- \ \ldots}{x^2+x}$$
$$=\lim_{x\to 0} \frac{2x-\frac{x^3}{6}+\frac{x^5}{120}- \ \ldots}{x^2+x}$$
$$=\lim_{x\to 0} \frac{2-\frac{x^2}{6}+\frac{x^4}{120}- \ \ldots}{x+1}=2.$$

2. Use a Maclaurin's series to evaluate $\sqrt{1.01}$ correct to six places of decimals.

   Solution

   We consider the expansion of the function $(1+x)^{\frac{1}{2}}$ and then substitute $x=0.01$
$$(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\ldots$$

That is,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ \ldots$$

Substituting $x = 0.01$ gives

$$\sqrt{1.01} = 1 + \frac{1}{2}\times 0.01 - \frac{1}{8}\times 0.0001 + \frac{1}{16}\times 0.000001 - \ldots$$

$$= 1 + 0.005 - 0.0000125 + 0.0000000625 - \ \ldots$$

The fourth term will not affect the sixth decimal place in the result given by the first three terms; and this is equal to 1.004988 correct to six places of decimals.

3. Assuming the Maclaurin's series for e^x and $\sin x$ and assuming that they may be multiplied together term-by-term, obtain the expansion of $e^x \sin x$ in ascending powers of x as far as the term in x^5.

Solution

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{120} + \ldots\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} + \ \ldots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \ \ldots$$

11.5.3 TAYLOR'S SERIES

A useful consequence of Maclaurin's series is known as **"Taylor's series"**.

One form of Taylor's series is as follows:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ \ldots$$

Proof:

To obtain this result from Maclaurin's series, we let $f(x+h) \equiv F(x)$.

Then,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0) + \ ..$$

But, $F(0) = f(h)$, $F'(0) = f'(h)$, $F''(0) = f''(h)$, $F'''(0) = f'''(h)$,. . . which proves the result.

Note: An alternative form of Taylor's series, often used for approximations, may be obtained by interchanging the symbols x and h.

That is,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ \ldots$$

EXAMPLE

Given that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use Taylor's series to evaluate $\sin(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{4}$ and $h = 0.01$

Solution

Using the sequence of derivatives as in the Maclaurin's series for $\sin x$, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \ldots\ldots$$

Subsituting $x = \frac{\pi}{4}$ and $h = 0.01$,

$$\sin\left(\frac{\pi}{4} + 0.01\right) = \frac{1}{\sqrt{2}}\left(1 + 0.01 - \frac{(0.01)^2}{2!} - \frac{(0.01)^3}{3!} + \ \ldots\right)$$

$$= \frac{1}{\sqrt{2}}(1 + 0.01 - 0.00005 - 0.000000017 + \ldots\ldots)$$

The fourth term does not affect the fifth decimal place in the sum of the first three terms; and so

$$\sin\left(\frac{\pi}{4} + 0.01\right) \simeq \frac{1}{\sqrt{2}} \times 1.00995 \simeq 0.71414$$