# "JUST THE MATHS"

# SLIDES NUMBER

# 11.3

# DIFFERENTIATION APPLICATIONS 3
# (Curvature)

## by


## A.J.Hobson

# UNIT 11.3 - DIFFERENTIATION APPLICATIONS 3

# CURVATURE

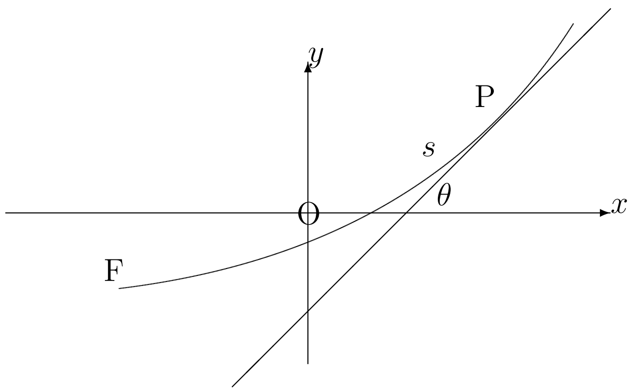
## 11.3.1 INTRODUCTION

The **"tightness of a bend"** on a curve is called its **"curvature"**.

Tight bends have large curvature.

Curves may be **"concave upwards"** ($\cup$), having positive curvature, or **"concave downwards"** ($\cap$), having negative curvature.

**DEFINITION**
Let $y = f(x)$ be the equation of a curve.




Let $\theta$ be the angle made with the positive $x$-axis by the tangent to the curve at a point, P$(x, y)$, on it.

Let $s$ be the distance to P, measured along the curve from some fixed point, F, on it.

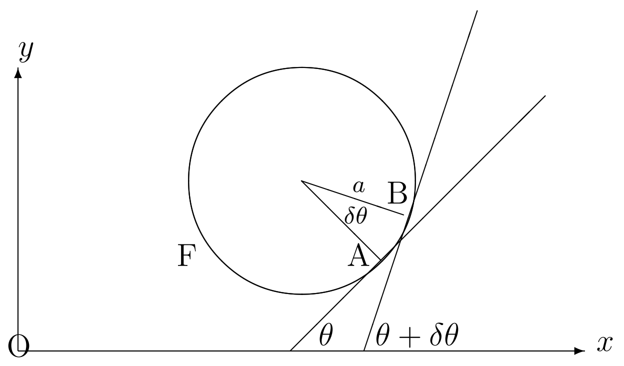
Then the curvature, $\kappa$, at P, is defined as the rate of increase of $\theta$ with respect to $s$.

$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

## EXAMPLE

Determine the curvature at any point of a circle with radius $a$.

## Solution




Let A be a point on the circle at which the tangent is inclined to the positive $x$-axis at an angle, $\theta$.

Let B be a point (close to A) at which the tangent is inclined to the positive $x$-axis at an angle, $\theta + \delta\theta$.

Let the length of the arc, AB, be $\delta s$, where distances, $s$, are measured along the circle in a counter-clockwise sense from the fixed point, F.

$\delta\theta$ is both the angle between the two tangents **and** the angle subtended at the centre of the circle by the arc, AB.

Thus,

$$\delta s = a\delta\theta,$$

or

$$\frac{\delta\theta}{\delta s} = \frac{1}{a}.$$

Allowing $\delta\theta$, and hence $\delta s$, to approach zero, we conclude that

$$\kappa = \frac{d\theta}{ds} = \frac{1}{a}.$$

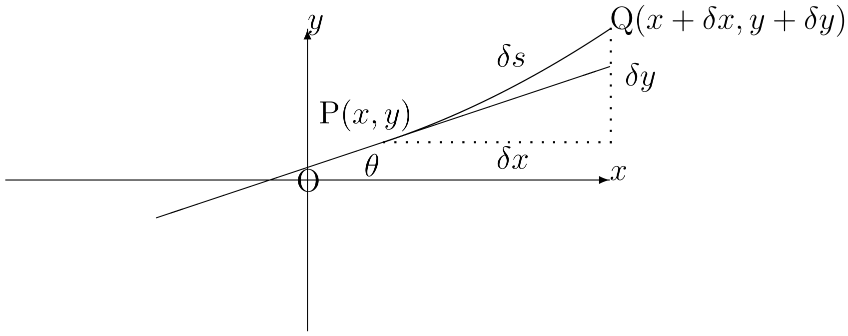
**Note:**
For the lower half of the circle, $\theta$ **increases** as $s$ increases, while, in the upper half of the circle, $\theta$ **decreases** as $s$ increases.

The curvature will, therefore, be positive for the lower half (which is concave upwards) and negative for the upper half (which is concave downwards).

**Summary**
The curvature at any point of a circle is numerically equal to the reciprocal of the radius.

## 11.3.2 CURVATURE IN CARTESIAN CO-ORDINATES

Given a curve whose equation is $y = f(x)$, let $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ be two neighbouring points on it (separated by a distance of $\delta s$ along the curve).




In this diagram,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \tan\theta$$

Also,

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lim_{\delta s \to 0} \frac{\delta x}{\delta s} = \cos\theta.$$

The curvature may therefore be evaluated as follows:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\cos\theta.$$

But,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{1}{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Finally,

$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}};$$

and so,

$$\kappa = \pm\frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}.$$

## Notes:

(i) For a curve which is concave upwards at a particular point, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **increase** as $x$ increases through the point.

Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be positive at the point.

(ii) For a curve which is concave downwards at a particular point, $\frac{\mathrm{d}y}{\mathrm{d}x}$, will **decrease** as $x$ increases through the point.

Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ will be negative at the point.

(ii) We may allow the value of the curvature to take the same sign as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Hence,

$$\kappa = \frac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2\right]^{\frac{3}{2}}}.$$

## EXAMPLE

Use the cartesian formula to determine the curvature at any point on the circle, centre $(0, 0)$ with radius $a$.

## Solution

The equation of the circle is

$$x^2 + y^2 = a^2.$$

For the upper half,

$$y = \sqrt{a^2 - x^2}.$$

For the lower half,

$$y = -\sqrt{a^2 - x^2}.$$

Considering the upper half,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{a^2 - x^2}}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2-x^2}}}{a^2 - x^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Therefore,

$$\kappa = \frac{-\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}}{\left(1+\frac{x^2}{a^2-x^2}\right)^{\frac{3}{2}}} = -\frac{a^2}{a^3} = -\frac{1}{a}.$$

Considering the lower half,

$$\kappa = \frac{1}{a}.$$