"JUST THE MATHS"

SLIDES NUMBER

11.2

DIFFERENTIATION APPLICATIONS 2
(Local maxima and local minima)
&
(Points of inflexion)

by

A.J.Hobson

UNIT 11.2 - APPLICATIONS OF DIFFERENTIATION 2

LOCAL MAXIMA, LOCAL MINIMA AND POINTS OF INFLEXION

11.2.1 INTRODUCTION

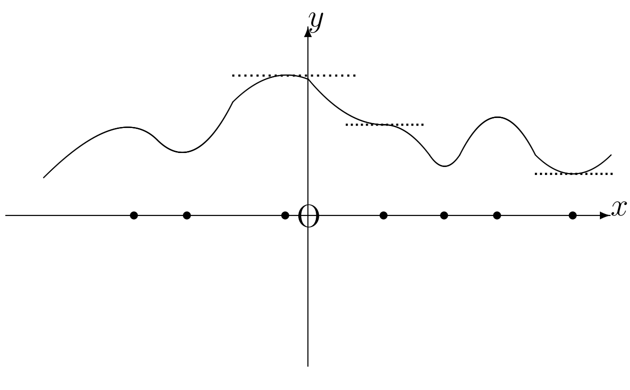
Any relationship,

$$y = f(x),$$

between two variable quantities, x and y, can usually be represented by a graph of y against x.

Any point (x_0, y_0) on the graph at which $\frac{dy}{dx}$ takes the value zero is called a **"stationary point"**.

The tangent to the curve at the point (x_0, y_0) will be parallel to x-axis.

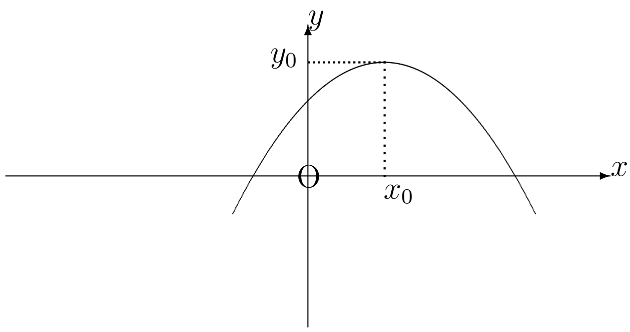


11.2.2 LOCAL MAXIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local maximum"** if y_0 is greater than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



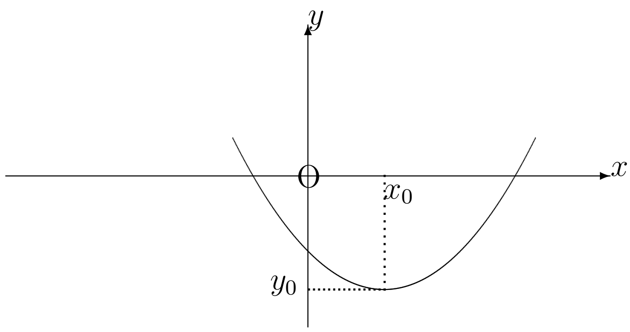
Note:
The definition of a local maximum point must refer to the behaviour of y in the **immediate neighbourhood** of the point.

11.2.3 LOCAL MINIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local minimum"** if y_0 is less than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



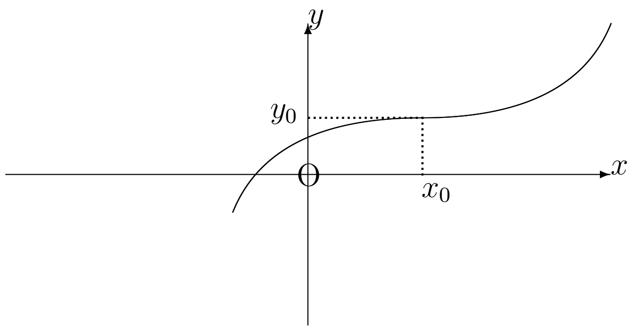
Note:
The definition of a local minimum point must refer to the behaviour of y in the **immediate neighbourhood** of the point.

11.2.4 POINTS OF INFLEXION

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"point of inflexion"** if the curve exhibits a change in the direction bending there.



11.2.5 THE LOCATION OF STATIONARY POINTS AND THEIR NATURE

First, we solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Having located a stationary point (x_0, y_0), we then determine whether it is a local maximum, local minimum, or point of inflexion.

METHOD 1. - The "First Derivative" Method

Let ϵ denote a number which is relatively small compared with x_0.

Examine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$, first at $x = x_0 - \epsilon$ and then at $x = x_0 + \epsilon$.

(a) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from positive to negative, there is a local maximum at (x_0, y_0).

(b) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from negative to positive, there is a local minimum at (x_0, y_0).

(c) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change, there is a point of inflexion at (x_0, y_0).

EXAMPLES

1. Determine the stationary point on the graph whose equation is

$$y = 3 - x^2.$$

Solution:

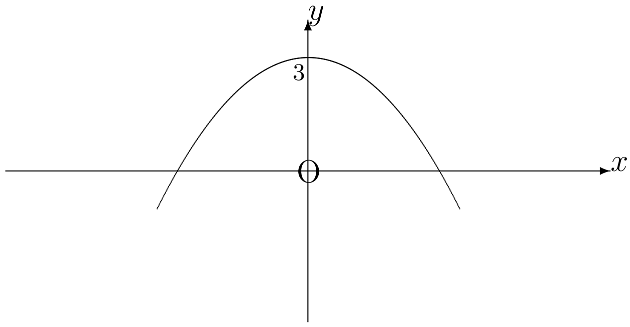
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$

which is equal to zero at the point where $x = 0$ and hence, $y = 3$.

If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.

If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.

Hence, there is a local maximum at the point $(0, 3)$.



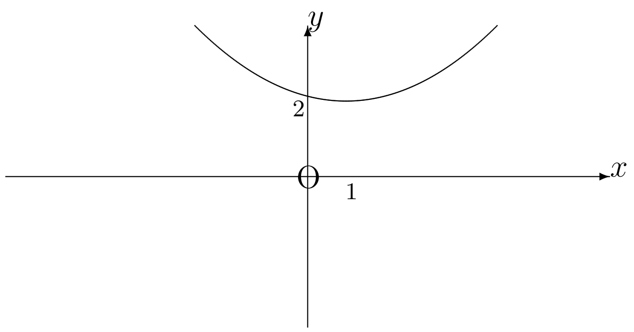
2. Determine the stationary point on the graph whose equation is
$$y = x^2 - 2x + 3.$$

Solution:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2,$$
which is equal to zero at the point where $x = 1$ and hence $y = 2$.

If $x = 1 - \epsilon$, (for example, $x = 1 - 0.01 = 0.99$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.

If $x = 1 + \epsilon$, (for example, $x = 1 + 0.01) = 1.01$, then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.

Hence there is a local minimum at the point $(1, 2)$.



3. Determine the stationary point on the graph whose equation is

$$y = 5 + x^3.$$

Solution:

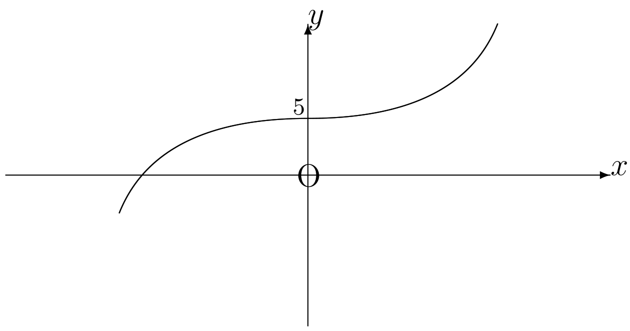
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

which is equal to zero at the point where $x = 0$ and hence, $y = 5$.

If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.

If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.

Hence, there is a point of inflexion at $(0, 5)$.

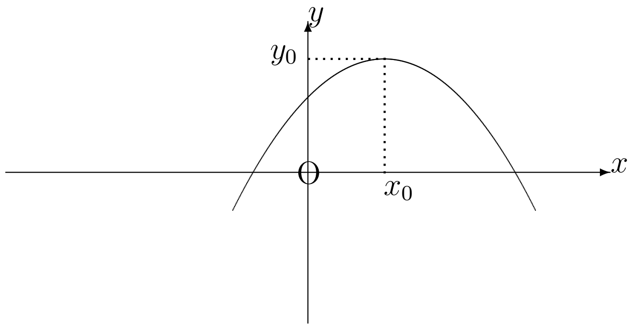


METHOD 2. - The "Second Derivative" Method

The graph of $\frac{dy}{dx}$ against x is called the **"first derived curve"**.

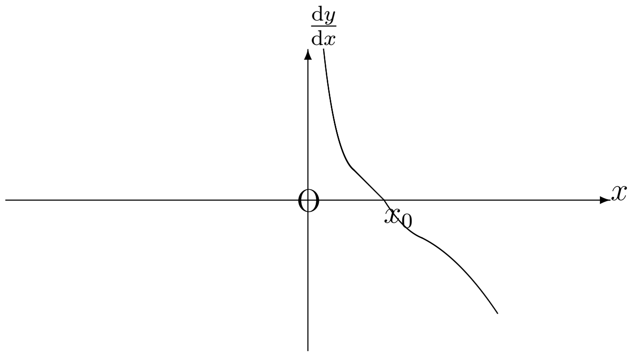
The properties of the first derived curve in the neighbourhood of a stationary point (x_0, y_0) may be used to predict the nature of this point.

(a) Local Maxima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ steadily decrease from large positive values to large negative values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going downwards"** tendency at $x = x_0$.



It may be expected that the slope at $x = x_0$ on the first derived curve is **negative**.

$$\text{TEST} \quad (\text{Max}) : \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0 \quad \text{at} \quad x = x_0.$$

EXAMPLE

For the curve whose equation is

$$y = 3 - x^2,$$

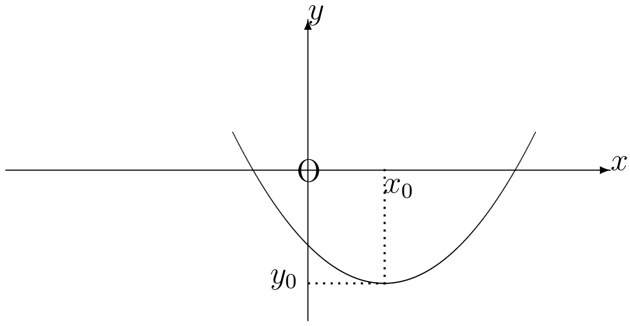
we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \quad \text{and} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2.$$

The second derivative is negative everywhere.

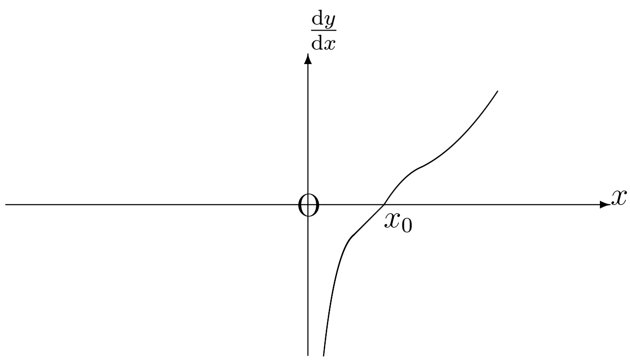
Hence, $(0, 3)$ (obtained earlier) is a local maximum.

(b) Local Minima



As x passes from values below x_0 to values above x_0 the corresponding values of $\frac{dy}{dx}$ steadily increase from large negative values to large positive values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going upwards"** tendency at $x = x_0$.



It may be expected that the slope at $x = x_0$ on the first derived curve is **positive**.

$$\text{TEST} \quad (\text{Min}) : \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0 \quad \text{at} \quad x = x_0.$$

EXAMPLE

For the curve whose equation is

$$y = x^2 - 2x + 3,$$

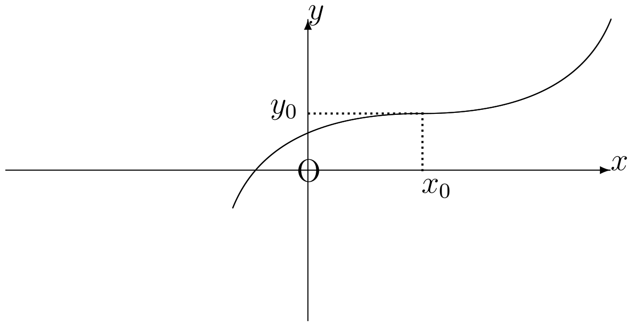
we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2 \quad \text{and} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2.$$

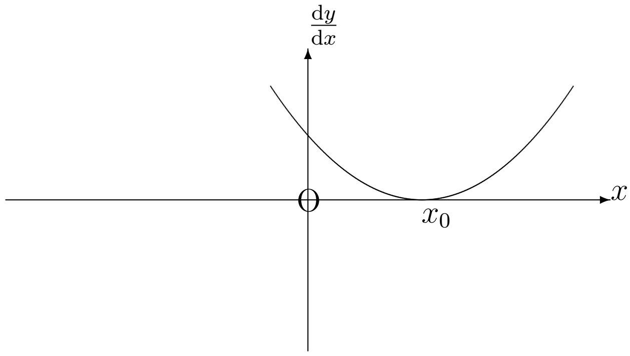
The second derivative is positive <u>everywhere</u>.

Hence, $(1, 2)$ (obtained earlier) is a local minimum.

(c) Points of inflexion



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ appear to reach either a minimum or a maximum value at $x = x_0$.



It may be expected that the slope at $x = x_0$ on the first derived curve is zero and changes sign as x passes through the value x_0.

$$\text{TEST (Infl)} : \frac{d^2y}{dx^2} = 0 \text{ at } x = x_0 \text{ and changes sign.}$$

EXAMPLE

For the curve whose equation is

$$y = 5 + x^3,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

The second derivative is zero when $x = 0$ and changes sign as x passes through the value zero.

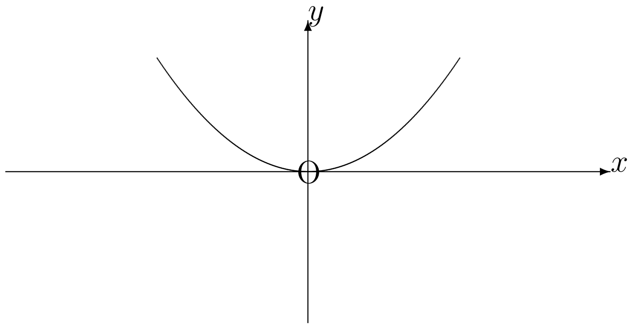
Hence the stationary point $(0, 5)$ (obtained earlier) is a point of inflexion.

Notes:
(i) For a stationary point of inflexion, it is not enough that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

without also the change of sign.

For example, $y = x^4$ has a local minimum at the point $(0, 0)$; but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$.



(ii) Some curves contain what are called **"ordinary points of inflexion"**.

They are not stationary points and hence, $\frac{dy}{dx} \neq 0$.

But we still use

$$\frac{d^2y}{dx^2} = 0 \text{ and changes sign.}$$

EXAMPLE

For the curve whose equation is

$$y = x^3 + x,$$

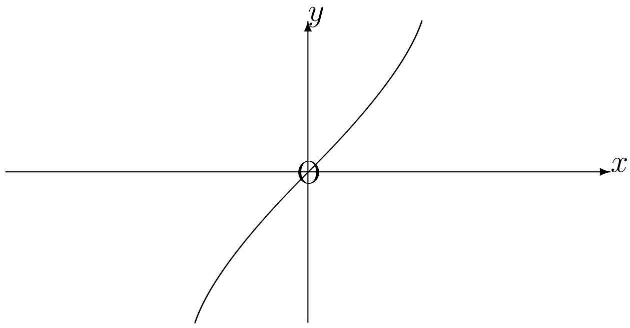
we have

$$\frac{dy}{dx} = 3x^2 + 1 \text{ and } \frac{d^2y}{dx^2} = 6x.$$

Hence, there are no stationary points at all.

But $\frac{d^2y}{dx^2} = 0$ at $x = 0$ and changes sign as x passes through $x = 0$.

That is, $y = x^3 + x$ has an ordinary point of inflexion at $(0, 0)$.



Note:

In any interval of the x-axis, the greatest value of a function of x will be either the greatest maximum or possibly the value at one end of the interval.

Similarly, the least value of the function will be either the smallest minimum or possibly the value at one end of the interval.