

“JUST THE MATHS”

SLIDES NUMBER

11.1

DIFFERENTIATION APPLICATIONS 1
(Tangents and normals)

by

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11.1.1 Tangents

11.1.2 Normals

UNIT 11.1 - APPLICATIONS OF DIFFERENTIATION 1

TANGENTS AND NORMALS

11.1.1 TANGENTS

The derivative of the function $f(x)$ can be interpreted as the gradient of the tangent to the curve $y = f(x)$ at the point (x, y) .

This information, together with the geometry of the straight line, determines the equation of the tangent to a given curve at a particular point on it.

EXAMPLES

1. Determine the equation of the tangent at the point $(-1, 2)$ to the curve whose equation is

$$y = 2x^3 + 5x^2 - 2x - 3.$$

Solution

$$\frac{dy}{dx} = 6x^2 + 10x - 2,$$

which takes the value -6 when $x = -1$.

Hence, the tangent is the straight line passing through the point $(-1, 2)$ having gradient -6 .

Its equation is therefore

$$y - 2 = -6(x + 1).$$

That is,

$$6x + y + 4 = 0.$$

2. Determine the equation of the tangent at the point $(2, -2)$ to the curve to the curve whose equation is

$$x^2 + y^2 + 3xy + 4 = 0.$$

Solution

$$2x + 2y \frac{dy}{dx} + 3 \left[x \frac{dy}{dx} + y \right] = 0.$$

That is,

$$\frac{dy}{dx} = -\frac{2x + 3y}{3x + 2y},$$

which takes the value -2 at the point $(2, -2)$.

Hence, the equation of the tangent is

$$y + 2 = -2(x - 2).$$

That is,

$$2x + y - 2 = 0.$$

3. Determine the equation of the tangent at the point where $t = 2$ to the curve given parametrically by

$$x = \frac{3t}{1+t} \quad \text{and} \quad y = \frac{t^2}{1+t}.$$

Solution

The point at which $t = 2$ has co-ordinates $(2, \frac{4}{3})$.

Furthermore,

$$\frac{dx}{dt} = \frac{3}{(1+t)^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{2t+t^2}{(1+t)^2}.$$

Thus,

$$\frac{dy}{dx} = \frac{2t+t^2}{3},$$

which takes the value $\frac{8}{3}$ when $t = 2$.

The equation of the tangent is

$$y - \frac{4}{3} = \frac{8}{3}(x - 2).$$

That is,

$$3y + 12 = 8x.$$

11.1.2 NORMALS

The normal to a curve at a point on it is defined to be a straight line passing through this point and perpendicular to the tangent there.

If the gradient of the tangent is m , then the gradient of the normal will be $-\frac{1}{m}$.

EXAMPLES

In the examples of section 11.1.1, the normals to each curve at the point given will have equations as follows:

1.

$$y - 2 = \frac{1}{6}(x + 1).$$

That is, $6y = x + 13$.

2.

$$y + 2 = \frac{1}{2}(x - 2).$$

That is, $2y = x - 6$.

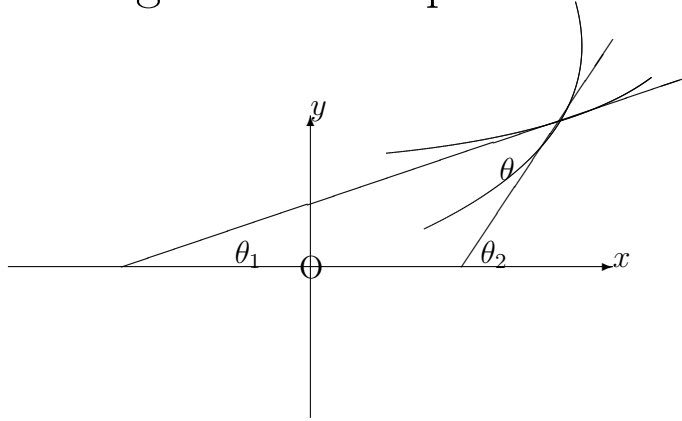
3.

$$y - \frac{4}{3} = -\frac{3}{8}(x - 2).$$

That is, $24y + 9x = 50$

The Angle between two curves.

The angle between two curves is defined to be the angle between the tangents at this point.



If the gradients of the tangents are $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$, the angle $\theta \equiv \theta_2 - \theta_1$ and is given by

$$\tan \theta = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1}.$$

That is,

$$\tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$